# เอกสารประกอบการสอน วิชา 2102202

# คณิตศาสตร์สำหรับวิศวกรรมไฟฟ้า II

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ภาควิชาวิศวกรรมไฟฟ้า
จุฬาลงกรณ์มหาวิทยาลัย

# Contents

# คำนำ

เอกสารประกอบการสอนวิชาคณิตศาสตร์สำหรับวิศวกรรมไฟฟ้าฉบับนี้ ได้ถูกเขียนขึ้นเพื่อใช้ประกอบการเรียนการสอนวิชา 2102202 (Electrical Engineering Mathematics II) อันเป็นหนึ่งในวิชาคณิตศาสตร์ที่เป็นวิชาบังคับ สำหรับนิสิตวิศวกรรมไฟฟ้า ปี 2 จุฬาลงกรณ์มหาวิทยาลัย ในวิชานี้เนื้อหาจะถูกแบ่งออกเป็น 2 ส่วนหลักคือ ด้านพีชคณิตเชิงเส้น (linear algebra) และการวิเคราะห์จำนวนเชิงซ้อน (complex analysis) จุดประสงค์หลักของการจัดทำเอกสารประกอบการสอนฉบับนี้ คือการสรุปใจความ และเลือกทฤษฎีบทที่สำคัญ มาจากตำรา H. Anton and C. Rorres. Elementary Linear Algebra with Supplemental Applications. Wiley, 10th edition, 2010. และ J. W. Brown and R. V. Churchill. Complex Variables and Applications. McGraw-Hill, 8th edition, 2009. อันเป็นสองตำราอ้างอิงหลักที่ใช้ประกอบกับวิชานี้ รวมถึงการบรรจุแบบฝึกหัดที่น่าสนใจที่ผู้เขียนได้จัดทำเอง หรือนำมาจากตำราเล่มอื่นๆ ในรายการหนังสืออ้างอิง เอกสารประกอบการสอนเล่มนี้ จึงมีไว้เพื่อให้นิสิตสามารถรับรู้ประเด็นสำคัญของวิชานี้ และติดตามเนื้อหากับตัวอย่างที่ไม่ยากนักจากห้องเรียน หลังจากนั้นนิสิตจึงสามารถนำไปอ่านรายละเอียด และทบทวนเนื้อหาจากตำราหลักได้ด้วยตนเอง โดยตรวจสอบจากรายการหนังสืออ้างอิงที่ระบุไว้ในตอนท้ายของแต่ละบท

เอกสารประกอบการสอนฉบับนี้ เริ่มจากการแนะนำการพิสูจน์ทางคณิตศาสตร์ (บทที่ 0) เพื่อให้นิสิตคุ้นเคยกับการอ่านและตีความข้อความทางคณิตศาสตร์ รวมไปถึงเทคนิคการพิสูจน์แบบต่างๆ อย่างไรก็ตาม เนื้อหาส่วนนี้ไม่ได้ถูกสอนในห้องโดยตรง แต่จะให้นิสิตไปค้นคว้ามาเอง และได้นำมาใช้จริงและอธิบายเพิ่มเติม เมื่อมีการพิสูจน์ทางคณิตศาสตร์ในเนื้อหาส่วนอื่น สำหรับเนื้อหาตั้งแต่บท 1-13 มีการใช้คำสำคัญต่างๆ เช่น นิยาม (definition) ซึ่งหมายถึง ข้อตกลงหรือการทำความเข้าใจ ในเรื่องที่จะกล่าวถึง, ทฤษฎีบท (theorem) อันแสดงถึงผลลัพธ์ที่เป็นสาระสำคัญของหัวข้อนั้นๆ หรือการใช้คำว่า ข้อเท็จจริง (fact) เพื่อแสดงถึง คุณสมบัติหรือผลลัพธ์ที่สามารถพิสูจน์ได้ จากการเรียนหัวข้อนั้นๆ ในเอกสารนี้ ผู้เขียนได้ใส่ไอคอน ✌ ไว้หน้าทฤษฎีบท หรือข้อเท็จจริง เพื่อแสดงว่า ผลลัพธ์นั้นๆ มีความสำคัญและจะพิสูจน์ในห้องเรียน และได้ใส่ไอคอน ✎ ไว้หน้าผลลัพธ์สำคัญที่ไม่ยากนักต่อการพิสูจน์ และต้องการให้นิสิตกลับไปพิสูจน์และทบทวนด้วยตนเอง หากเนื้อหาในบทใดที่เกี่ยวข้องกับการคำนวณด้วยคอมพิวเตอร์ได้ จะมีตัวอย่างการใช้ฟังก์ชันใน MATLAB เพื่อให้นิสิตไปตรวจสอบผลและทำการบ้านได้ในภายหลัง ในตอนท้ายของเอกสารนี้ ผู้เขียนได้รวบรวมตัวอย่างข้อสอบ จากการสอบย่อย สอบกลางภาค และสอบปลายภาค จาก 3 ภาคการศึกษา เพื่ออ้างอิงถึงการประเมินความรู้ความเข้าใจของนิสิตในวิชานี้

เอกสารนี้ ในส่วนที่เป็นพีชคณิตเชิงเส้น ได้ถูกนำมาใช้ประกอบการสอนเป็นเวลา 3 ภาคการศึกษา และในส่วนที่เป็นการวิเคราะห์จำนวนเชิงซ้อนได้ถูกนำมาใช้เป็นเวลา 2 ภาคการศึกษา เมื่อจบแต่ละภาคการศึกษา ผู้เขียนได้แก้ไขเอกสารมาอย่างต่อเนื่องตามผลลัพธ์ที่ได้จากการประเมินความเข้าใจของนิสิตในห้องเรียน รวมถึงจากความเห็นของอาจารย์ผู้ร่วมสอน ผู้เขียนจึงต้องขอขอบคุณ ผศ. ดร.สุชิน อรุณสวัสดิ์วงศ์ และ รศ.ดร. นิศาชล ตั้งเสงี่ยมวิสัย สำหรับความเห็นที่เป็นประโยชน์ต่อการสอนและการจัดทำเอกสารประกอบการสอน มา ณ ที่นี้


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# บทที่ 1

# Introduction to Mathematical Proofs

บทนี้จะแนะนำการอ่านและตีความข้อความทางคณิตศาสตร์ การรู้จักถึงเงื่อนไขจำเป็นและเพียงพอ และเทคนิคการพิสูจน์ข้อความทางคณิตศาสตร์ด้วยวิธีต่างๆ

EE202 - EE MATH II Jitkomut Songsiri

# 1. Mathematical Proofs

- conditional statements
- sufficient and necessary conditions
- methods of proofs
- disproving statements
- proofs of quantified statements



## Statements

a **statement** is a declarative sentence that is true *or* false but not both

examples:

- $3 + 4 = 7$
- $5 \cdot 2 - 3 = 9$
- if $x$ is an integer, then $2x$ is an even integer

the following sentences are not statements

- Bangkok is a lovely city (it's a matter of opinion)
- $2x - 3 = 4$ (we do not know what $x$ is)



## Conditional statements

for statements $P$ and $Q$, a **conditional statement** is the statement:

**If** $P$, **then** $Q$

and is denoted by $P \Rightarrow Q$ (also stated as $P$ implies $Q$)

example: 'if students obtain a score higher than 80 then they will get an A'

**truth table**

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

$P \Rightarrow Q$ is logically equivalent to

- $\neg P \vee Q$
- $\neg Q \Rightarrow \neg P$

**beware !** $P \Rightarrow Q$ is NOT logically equivalent to $Q \Rightarrow P$

## Biconditional statements

the conjunction of a conditional statement and its *converse*:

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called the **biconditional** of $P$ and $Q$, which is expressed as

$P$ **if and only if** $Q$

and denoted by $P \Leftrightarrow Q$

**truth table**

| $P$ | $Q$ | $P \Leftrightarrow Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

examples:

- $x = 2$ if and only if $3x = 6$
- $|x| = 4$ if and only if $x^2 = 16$

$P \Leftrightarrow Q$ is true only when $P$ and $Q$ have the same truth values



## Sufficient and Necessary conditions

consider a (true) conditional statement: $P \Rightarrow Q$, we say

- $P$ is **sufficient** for $Q$
- $Q$ is **necessary** for $P$
- $P$ **only if** $Q$

example: if $x = -3$ then $|x| = 3$ (a true conditional statement)

- '$P$ is sufficient for $Q$' means
  the truth of $x = -3$ is sufficient for concluding the truth of $|x| = 3$
- '$P$ only if $Q$' and '$Q$ is necessary for $P$' have the same meaning:
  $x = -3$ is *true only* under the condition that $|x| = 3$ (because if $|x| \neq 3$ then $x = -3$ can't be true)



however, $|x| = 3$ is *not a sufficient condition* for $x = -3$

(because if $|x| = 3$ then $x$ can be either $3$ or $-3$)

$i.e.$, the converse of 'if $x = -3$ then $|x| = 3$' is false

consider a (true) biconditional statement: $P \Leftrightarrow Q$, we say

$P$ is **sufficient** and **necessary** for $Q$

example: $|x| = 2$ if and only if $x^2 = 4$ (a true biconditional statement)

- saying $|x| = 2$ is equivalent to saying $x^2 = 4$

**more examples:**

- being at least $18$ years old is *necessary* for applying a driver license
  $i.e.$,
  - if you're a driver, everyone knows you must be at least $18$ years old
  - if you're younger than $18$ then you can't have a driver license
- if a person holds the title 'Miss Thailand' then that person must be 1) female 2) adult and 3) unmarried
  $i.e.$,
  - stating that 'Jenny is Miss Thailand' is *sufficient* to know that she is female and she must be old enough (an adult)
  - being unmarried is a *necessary* condition for being Miss Thailand because if a woman is married, she can't apply for this position



## Mathematical terminology

- an **axiom** is a math statement that is *self-evidently true* w/o proof
- a **definition** is an *agreement* as to the meaning of a particular term
- a **proof** is a sequence of math arguments demonstrating the truth of given results
- a **theorem** or a **proposition** is any mathematical statement that can be shown to be true using accepted logical and mathematical arguments
- a **lemma** is a true mathematical statement that was proven mainly to help in the proof of some theorem
- a **corollary** is used to refer to a theorem that is easily proven once some other theorem has been proven



## Direct proofs

a **direct proof** of $P \Rightarrow Q$ typically consists of these steps:

1. start from assuming $P$ is true then
2. develop a set of logical arguments to conclude $Q$

**example:** show that if $x, y \in \mathbf{R}$ then $x^2 + y^2 \geq |xy|$
Proof. let $x, y \in \mathbf{R}$ and consider $(|x| - |y|)^2$

$$(|x| - |y|)^2 = |x|^2 + |y|^2 - 2|xy|$$

since the LHS is nonnegative, it follows that

$$(|x| - |y|)^2 = x^2 + y^2 - 2|xy| \geq 0$$

and hence $x^2 + y^2 \geq 2|xy| \geq |xy|$ □

## Proof by contrapositive

a **contrapositive proof** of a statement $P \Rightarrow Q$ uses the fact that

$$P \Rightarrow Q \quad \text{is logically equivalent to} \quad \neg Q \Rightarrow \neg P$$

so we can use a direct proof to show that $\neg Q \Rightarrow \neg P$ is true

**example:** let $x \in \mathbf{R}$. show that if $x^2 + 2x < 0$ then $x < 0$

Proof. we will show that if $x \geq 0$ then $x^2 + 2x \geq 0$

- if $x \geq 0$ then obviously $2x \geq 0$
- $x^2$ is always nonnegative

therefore, the sum of $x^2$ and $2x$ is nonnegative, finishing the proof □



## Proof by contradiction

**idea:** $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$, so if we do as follows:

1. assume $P$ is true (accept all the hypotheses) and $Q$ is false (negate the conclusion)
2. try to prove that this leads to a **contradiction**

then we have shown that $\neg(P \Rightarrow Q)$ is false or that $P \Rightarrow Q$ is true

**example:** show that if $n$ is an even integer then so is $n^2$

Proof. assume $n$ is even but $n^2$ is not

since $n$ is even, we can express $n = 2k$ where $k$ is some positive integer

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

since $2k^2$ is also an integer, $n^2$ must be also even, which is a contradiction



## Proof by induction

**principle of mathematical indunction** states that

the statement $P(n)$ is true for all $n \in \mathbf{N}$ if

1. $P(1)$ is true
2. for each $k \in \mathbf{N}$, if $P(k)$ is true then $P(k+1)$ is also true

**example:** show that $\sum_{i=1}^{n} i = n(n+1)/2$ for $n = 1, 2, \ldots$

Proof. let $P(n)$ be the statement $\sum_{i=1}^{n} i = n(n+1)/2$

- $P(1)$ is true because $1 = 1 \cdot (1+1)/2$
- assume $P(k)$ is true and show that $P(k+1)$ is true:

$$\sum_{i=1}^{n+1} i = n + 1 + \sum_{i=1}^{n} i = n + 1 + n(n+1)/2 = (n+1)(n+2)/2$$

## Disproving statements

a **conjecture** is any math statement that has *not* been proved or disproved

**disproving** a conjecture requires only a *single example* to show the conjecture is *false*

such example is called a **counterexample**

**example:** $(x+y)^2 = x^2 + y^2$ for all $x, y \in \mathbf{R}$ (conjecture)

$x = 1, y = 1$ is a counterexample that disproves the conjecture because

$$(1+1)^2 = 4 \neq 1^2 + 1^2 = 2$$

(because the conjecture says the identity holds *for all* $x, y$, we just gave a value of $x, y$ that disproves it)



**example:** let $A$ be a square matrix. if $A^2 = I$ then $A = I$ or $-I$

the conjecture is false because if we consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then we can verify that

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

hence, $A^2 = I$ does not necessarily imply that $A = I$ or $A = -I$

but $A$ could be other matrices (at least the counterexample we just gave)



## Quantifiers

- the quantifying clause '**for every, for all, for each**' is denoted by $\forall$
- the quantifying clause '**there exists, there is some**' is denoted by $\exists$
- $x \in S$ means '$x$ is a member of set $S$' or '$x$ belongs to $S$'

examples:

- for every positive real number $x$, $x^3 - 2x^2 + x > 0$

$$\forall x \in \mathbf{R}, \quad x^3 - 2x^2 + x > 0$$

- there exists a real number $x$ such that $x^2 - 2x = 4$

$$\exists x, \quad x^2 - 2x = 4$$

## Proofs of quantified statements

**statements containing 'for some' or 'there exists'**

**example:** prove or disprove '$\exists A \in \mathbf{R}^{2\times 2},\ \det(A) = 1$'

to prove that it's true, we just need to come up with *an example* of $A$:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and show that} \quad \det(A) = 1$$

hence, the statement is true

**example:** prove or disprove '$\exists x \in \mathbf{R},\ x^4 + 2x^2 + 1 = 0$'

if $x \in \mathbf{R}$, then $x^4 \geq 0$ and $x^2 \geq 0$, so $x^4 + 2x^2 + 1 \geq 1$

$x^4 + 2x^2 + 1$ can't be $0$ **for any** $x \in \mathbf{R}$, so the statement is false

- proving that the statement is true is typically (but not always) *simple*
- disproving the statement may require some effort



**statements containing 'for all' or 'for any'**

**example:** prove or disprove '$\forall x, y \in \mathbf{R},\ |x + y| \leq |x| + |y|$'

$$(x+y)^2 = x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|xy| = (|x| + |y|)^2$$

so the statement is true

**example:** prove or disprove '$AB = BA$ for any square matrices $A, B$'

disproving it is easy because we can just give an example of $A, B$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and show that $AB = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (so the statement is false)

- proving the statement is true may require some effort
- disproving the statement is typically *easy* (by giving a counterexample)



## Common mistakes

**example:** show that for any $\alpha \in \mathbf{R}, A \in \mathbf{R}^{n\times n},\ \det(\alpha A) = |\alpha|^n \det A$

one may show as follows

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \Longrightarrow \quad \det(A) = 5 \text{ and } \det(\alpha A) = \begin{vmatrix} \alpha & 2\alpha \\ -\alpha & 3\alpha \end{vmatrix} = 5\alpha^2$$

so $\det(\alpha A) = \alpha^2 \det(A)$ as desired

the above argument *cannot* be a proof because we just showed for *one* particular value of $A$

in fact, we have to show that the statement is true **for all** square matrices

**example:** show that for any $x, y \in \mathbf{R}$, $(x+y)^2 \leq 2(x^2+y^2)$

if one writes an argument like this:

$$x^2 + 2xy + y^2 \leq 2x^2 + 2y^2 \;\Rightarrow\; x^2 + y^2 - 2xy \geq 0 \;\Rightarrow\; (x-y)^2 \geq 0$$

then it can't be a proof because:

- we can't start a proof from the result we're going to prove !
- each step of argument must be explained with logical reasoning
- a good proof must be clear by itself; always explain with details
- the lastly obtained result must conclude what you want to prove



example of proof: for any $x, y \in \mathbf{R}$, $(x-y)^2$ is always nonnegative

- expanding $(x-y)^2$ gives

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

- add $x^2 + 2xy + y^2$ on both sides

$$x^2 + 2xy + y^2 \leq 2x^2 + 2y^2$$

- complete the square and we finish the proof

$$(x+y)^2 \leq 2(x^2+y^2)$$



## References

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# บทที่ 2

# System of linear equations

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถวิเคราะห์และจำแนกได้ว่า สมการทั่วไปที่ให้มาเป็นสมการเชิงเส้นหรือไม่
- สามารถจัดรูปแบบสมการเชิงเส้นให้อยู่ในรูปแบบสมการเมทริกซ์ได้
- สามารถวิเคราะห์ได้ว่าสมการเชิงเส้นหนึ่งๆ มีคำตอบหรือไม่ ถ้าหากมี จะมีคำตอบเพียงหนึ่งเดียวหรือไม่
- สามารถใช้การดำเนินการตามแถวนอนชั้นมูลฐาน (elementary row operations) มาใช้ในการแก้สมการเชิงเส้นได้

# 2. System of Linear Equations

- linear equations
- elementary row operations
- Gaussian elimination



## Linear equations

a general linear system of $m$ equations with $n$ variables is described by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{ij}, b_j$ are constants and $x_1, x_2, \ldots, x_n$ are unknowns

- equations are linear in $x_1, x_2, \ldots, x_n$
- existence and uniqueness of a solution depend on $a_{ij}$ and $b_j$



## Example: solving ordinary differential equations

given $y(0) = 1, \dot{y}(0) = -1, \ddot{y}(0) = 0$, solve

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 0$$

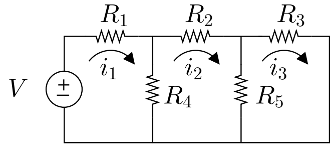
the closed-form solution is

$$y(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{-3t}$$

$C_1, C_2$ and $C_3$ can be found by solving a set of linear equations

$$\begin{aligned} 1 &= y(0) &&= C_1 + C_2 + C_3 \\ -1 &= \dot{y}(0) &&= -C_1 - 2C_2 - 3C_3 \\ 0 &= \ddot{y}(0) &&= C_1 + 4C_2 + 9C_3 \end{aligned}$$

## Example: linear static circuit




given $V$, $R_1, R_2, \ldots, R_5$, find the currents in each loop

by KVL, we obtain a set of linear equations

$$\begin{aligned} V &= (R_1 + R_4)i_1 - R_4 i_2 \\ 0 &= -R_4 i_1 + (R_2 + R_4 + R_5)i_2 - R_5 i_3 \\ 0 &= -R_5 i_2 + (R_3 + R_5) i_3 \end{aligned}$$

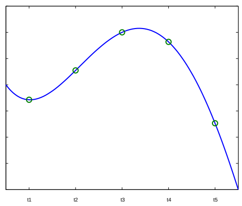


## Example: polynomial interpolation

fit a polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

through $n$ points $(t_1, y_1), \ldots, (t_n, y_n)$



problem data (parameters): $t_1, \ldots, t_n, y_1, \ldots, y_n$

problem variables: find $x_1, \ldots, x_n$ such that $p(t_i) = y_i$ for all $i$



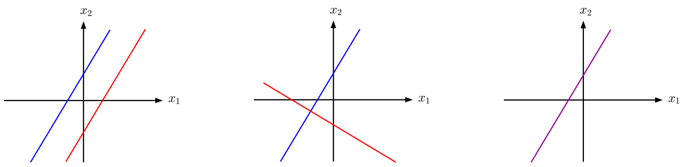
write out the conditions on $x$:

$$\begin{aligned} p(t_1) &= x_1 + x_2 t_1 + x_3 t_1^2 + \cdots + x_n t_1^{n-1} &= y_1 \\ p(t_2) &= x_1 + x_2 t_2 + x_3 t_2^2 + \cdots + x_n t_2^{n-1} &= y_2 \\ &\vdots & \vdots \\ p(t_n) &= x_1 + x_2 t_n + x_3 t_n^2 + \cdots + x_n t_n^{n-1} &= y_n \end{aligned}$$

## Special case: two variables

**Examples:**

$$\begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= -2 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ x_1 + x_2 &= -1 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= -2 \end{aligned}$$




(a) no solution (b) one solution (c) infinitely many solutions

- no solution if two lines are parallel but different interceptions on $x_2$-axis
- many solutions if the two lines are identical



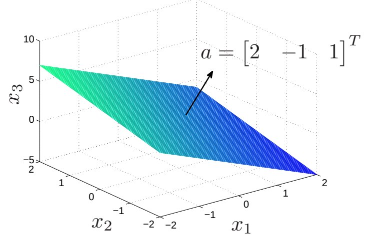
## Geometrical interpretation

the set of solutions to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

can be interpreted as a hyperplane on $\mathbf{R}^n$

$$2x_1 - x_2 + x_3 = 1$$




a solution to $m$ linear equations is an **intersection** of $m$ hyperplanes



## Existence and uniqueness of solutions

**existence:**

- no solution
- a solution exists

  **uniqueness:**

  – the solution is unique
  – there are infinitely many solutions

every system of linear equations has zero, one, or infinitely many solutions

there are no other possibilities

**no solution**

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= 2 \end{aligned}$$

**unique solution**

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= -2 \end{aligned}$$

**infinitely many solutions**

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + x_3 &= -1 \\ 3x_1 - 2x_2 + 3x_3 &= 3 \end{aligned}$$



## Elementary row operations

define the **augmented matrix** of the linear equations on page 2-2 as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

the following operations on the row of the augmented matrix:

1. multiply a row through by a nonzero constant
2. interchange two rows
3. add a constant times one row to another

do not alter the solution set and yield a simpler system

these are called **elementary row operations** on a matrix



**example:**

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \overset{\text{augmented matrix}}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add the first row to the second ($R_1 + R_2 \to R_2$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add $-2$ times the first row to the third ($-2R_1 + R_3 \to R_3$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ -7x_2 - 6x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

multiply the second row by $1/4$ $(R_2/4 \to R_2)$

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 2 \\ x_2 + \frac{3}{4}x_3 &=& \frac{1}{4} \\ -7x_2 - 6x_3 &=& -1 \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

add 7 times the second row to the third $(7R_2 + R_3 \to R_3)$

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 2 \\ x_2 + \frac{3}{4}x_3 &=& \frac{1}{4} \\ -\frac{3}{4}x_3 &=& \frac{3}{4} \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & -3/4 & 3/4 \end{bmatrix}$$

multiply the third row by $-4/3$ $(-4R_3/3 \to R_3)$

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 2 \\ x_2 + \frac{3}{4}x_3 &=& \frac{1}{4} \\ x_3 &=& -1 \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



add $-3/4$ times the third row to the second $(R_2 - (3/4)R_3 \to R_2)$

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 &=& 2 \\ x_2 &=& 1 \\ x_3 &=& -1 \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add $-3$ times the second row to the first $(R_1 - 3R_2 \to R_1)$

$$\begin{array}{rcl} x_1 + 2x_3 &=& -1 \\ x_2 &=& 1 \\ x_3 &=& -1 \end{array} \implies \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add $-2$ times the third row to the first $(R_1 - 2R_2 \to R_1)$

$$\begin{array}{rcl} x_1 &=& 1 \\ x_2 &=& 1 \\ x_3 &=& -1 \end{array} \implies \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



## Gaussian Elimination

- a systematic procedure for solving systems of linear equations
- based on performing row operations of the augmented matrix

**Definition:** a matrix is in **row echelon form** if

1. a row does not consist entirely of zeros, then the first nonzero number in the row is a 1 (called a leading 1)
2. all nonzero rows are above any rows of all zeros
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row

**examples:**

$$\begin{bmatrix} 1 & 4 & -3 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Definition:** a matrix is in **reduced row echelon form** if

- it is in a row echelon form and
- every leading 1 is the only nonzero entry in its colum

**examples:**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



## Facts about echelon forms

1. every matrix has a *unique* reduced row echelon form
2. row echelon forms are not unique
3. all row echelon forms of a matrix have the same number of zero rows
4. the leading 1's always occur in the same positions in the row echelon forms of a matrix $A$

   those positions are callled the **pivot** positions of $A$

   a column that contains a pivot position is called a **pivot column** of $A$.



## Inspecting a solution

- simplify the augmented matrix to the *reduced echelon form*
- read the solution from the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies 0 \cdot x_3 = 1 \quad \text{(no solution)}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \implies x_1 = -2, \;\; x_2 = -1, \;\; x_3 = 5 \quad \text{(unique solution)}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies x_1 = 2, \; x_2 = 1 \quad \text{(unique solution)}$$

another example

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{rcl} x_1 + 3x_2 & = & -2 \\ x_2 - x_3 & = & 1 \end{array}$$

**Definition:**

- the corresponding variables to the leading 1's are called **leading variables**
- the remaining variables are called **free variables**

here $x_1, x_2$ are leading variables and $x_3$ is a free variable

let $x_3 = t$ and we obtain

$$x_1 = -3t - 2, \quad x_2 = t + 1, \quad x_3 = t$$

(many solutions)



$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies x_1 - 5x_2 + x_3 = 4$$

$x_1$ is the leading variable, $x_2$ and $x_3$ are free variables

let $x_2 = s$ and $x_3 = t$ we obtain

$$\begin{array}{rcl} x_1 & = & 5s - t + 4 \\ x_2 & = & s \\ x_3 & = & t \end{array}$$

(many solutions)

by assigning values to $s$ and $t$, a set of parametric equations:

$$\begin{array}{rcl} x_1 & = & 5s - t + 4 \\ x_2 & = & s \\ x_3 & = & t \end{array}$$

is called a **general solution** of the system



## Gaussian-Jordan elimination

- simplify an augmented matrix to the reduced row echelon form
- inspect the solution from the reduced row echelon form
- the algorithm consists of two parts:
  - **forward phase:** zeros are introduced below the leading 1's
  - **backward phase:** zeros are introduced above the leading 1's

**example:**

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 & = & 8 \\ -x_1 - 2x_2 + 3x_3 & = & 1 \\ 3x_1 - 7x_2 + 4x_3 & = & 10 \end{array} \implies \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

use row operations

$$\begin{matrix} R_1 + R_2 \to R_2 & -3R_1 + R_3 \to R_3 & (-1)\cdot R_2 \to R_2 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} 10R_2 + R_3 \to R_3 & R_3/(-52) \to R_3 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{matrix}$$

(a row echelon form)

we have added zero below the leading 1's (forward phase)



continue performing row operations

$$\begin{matrix} 5R_3 + R_2 \to R_2 & -R_2 + R_1 \to R_1 & -2R_3 + R_1 \to R_1 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{matrix}$$

(reduced echelon form)

we have added zero above the leading 1's (backward phase)

the system has a unique solution

$$x_1 = 3, \quad x_2 = 1, \quad x_3 = 2$$



## Homogeneous linear systems

**Definition:**

a system of linear equations is said to be **homogeneous** if $b_j$'s are all zero

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \quad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- $x_1 = x_2 = \cdots = x_n = 0$ is the **trivial** solution
- if $(x_1, x_2, \ldots, x_n)$ is a solution, so is $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ for any $\alpha \in \mathbf{R}$
- hence, if a solution exists, then the system has infinitely many solutions (by varying $\alpha$)

more properties

- the last column of the augmented matrix is entirely zero
- the zero columns do not alter under any row operations, so the linear systems corresponding to the reduced echelon form is homogeneous
- if the reduced row echelon form has $r$ *nonzero* rows, then the system has $n - r$ free variables
- a homogeneous linear system with more unknowns than equations has infinitely many solutions



**example**

$$\begin{array}{rcl} x_1 - x_2 + 2x_3 - x_4 & = & 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 & = & 0 \\ -x_1 + 2x_2 - 4x_3 + x_4 & = & 0 \\ 3x_1 - 3x_4 & = & 0 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1 & -1 & 2 & -1 & 0 \\ 2 & 1 & -2 & -2 & 0 \\ -1 & 2 & -4 & 1 & 0 \\ 3 & 0 & 0 & -3 & 0 \end{bmatrix}$$

the reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{array}{rcl} x_1 - x_4 & = & 0 \\ x_2 - 2x_3 & = & 0 \end{array}$$

define $x_3 = s, x_4 = t$, the parametric equation is

$$x_1 = t, \quad x_2 = 2s, \quad x_3 = s, \quad x_4 = t$$

there are two nonzero rows, so we have two $(n - 2 = 2)$ free variables



## MATLAB commands

`rref(A)` produces the reduced row echelon form of a matrix $A$

```
>> A = [-1 2 4 1;0 1 2 1;2 3 6 5]
A =
    -1     2     4     1
     0     1     2     1
     2     3     6     5

>> rref(A)

ans =

     1     0     0     1
     0     1     2     1
     0     0     0     0
```

## References

Chapter 1 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

## Exercises

1. Consider the system of linear equations:

$$\begin{aligned} 3x_1 - 4x_2 + x_3 + 3x_4 &= b_1 \\ -x_1 - 3x_2 - 4x_3 - 4x_4 &= b_2 \\ -2x_1 + 4x_2 - 2x_3 - 4x_4 &= b_3 \\ -x_1 + 4x_2 - x_3 - 3x_4 &= b_4 \end{aligned}$$

and denote $b = (b_1, b_2, b_3, b_4)$. In this problem, you are about to solve the linear equations for seven values of $b$. You should not resolve the equations every time the new vector $b$ is given. Propose an efficient way to find the solutions without repeating the process of performing elementary row operations.

(a) Denote the standard unit vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solve the linear equations with $b = \mathbf{e}_i$ for $i = 1, 2, 3, 4$. If the linear system has a solution for all the four choices of $b$, refer to the those solutions as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

(b) Consider the matrix

$$A = \begin{bmatrix} 3 & -4 & 1 & 3 \\ -1 & -3 & -4 & -4 \\ -2 & 4 & -2 & -4 \\ -1 & 4 & -1 & -3 \end{bmatrix}.$$

How does $A$ relate to the augmented matrix of the linear system ? From the solutions $\mathbf{x}_i$ in part a), construct the following matrix

$$B = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}.$$

Compute $BA$ and explain what you found.

(c) Explain how you would apply the result in part b) to solve the linear system with the following values of $b$.

$$b = (5, 5, -4, 3), \quad b = (20, 17, -18, 16), \quad b = (-3, 15, 6, 3).$$

2. Given the following five data points $(x_i, y_i)$, $i = 1, 2, \ldots, 5$ as follows.

$$(-2, -5.1), \quad (-1, -0.5), \quad (0, 0.5), \quad (1, 0.9), \quad (2, 1.3)$$

These points are plotted in Figure 2. In practice, we wish to explain a relationship between $x_i$ and $y_i$ through a function $y = f(x)$. Therefore, the goal is build a curve $f(x)$ that exactly passes these points (if possible) or the curve should be as close to these data points as possible. In this problem, we specifically choose a polynomial function of order $4$:

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

(a) Explain if we can find the above polynomial that passes the given five points exactly. If it is possible, give the expression of $f(x)$.

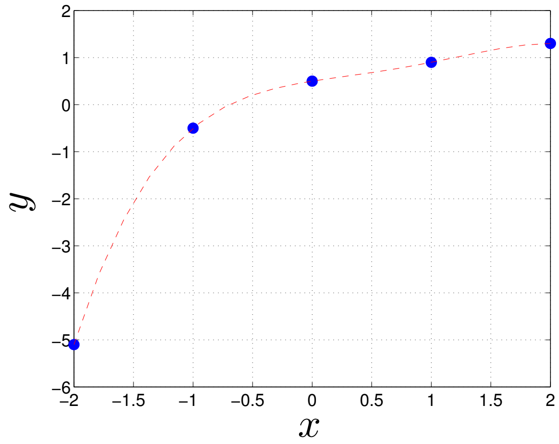



Figure 2.1: Fitting a $4$th-order polynomial to five data points.

(b) If one more data point is added, are we still able to find a $4$th-order polynomial that passes all the six points ? Note that in this case, we have $6$ data points, but only have $5$ parameters to be identified. Do you think it depend on the value of the additional point ? Justify your answer.

(c) Verify the part b) with $(x_6, y_6) = (3, 1)$ and $(x_6, y_6) = (3, 1.3)$.

3. True/False questions. For each of the following statements, either show that it is true, or give a specific counterexample if it is false.

(a) Consider the system of two equations.

$$5u + 2\log v = 16, \qquad -u - 3\log v = 4$$

By introducing some new variables, we can use row operations to solve this system.

(b) The system of equations:

$$(x-2)(y-5) = 3, \qquad 6x/y = 5 - x$$

is linear in $x$ and $y$.

(c) Every system of two equations with two unknown variables has a unique solution.

(d) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a reduced row echelon matrix.

(e) $\begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a reduced row echelon matrix.

(f) $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a reduced row echelon matrix.

(g) Every equation of the form $a_1x_1 + a_2x_2 = b$ has at least one solution for any nonzero $a_1, a_2, b$.

(h) For any nonzero $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, if $m > n$ then the system $Ax = b$ has no solution.

(i) For any $A \in \mathbb{R}^{m\times n}$, the system $Ax = 0$ always has infinitely many solutions.

(j) An augmented matrix for the equation $a_1x_1 + a_2x_2 = b$ is

$$\left[\begin{array}{c|c} a_1 & b \\ a_2 & b \end{array}\right]$$

(k) If

$$\left[\begin{array}{ccc|c} 2 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right]$$

is an augmented matrix of a system of linear equations, the solution set is all the vectors that are multiple of $(-4, 1, 2)$.

4. Consider a system of linear equations with variables $x_1, x_2, x_3$ and $x_4$.

$$\begin{aligned} x_1 + ax_2 + a^2x_3 + a^3x_4 &= 0 \\ x_1 + bx_2 + b^2x_3 + b^3x_4 &= 0 \\ x_1 + cx_2 + c^2x_3 + c^3x_4 &= 0 \\ x_1 + dx_2 + d^2x_3 + d^3x_4 &= 0 \end{aligned} \tag{2.1}$$

The coefficients $a, b, c$ and $d$ are all nonzero. Moreover, any two coefficients are not equal, i.e.,

$$a \neq b, \quad a \neq c, \quad a \neq d, \quad b \neq c, \quad b \neq d, \quad c \neq d.$$

(a) Reduce the augmented matrix for the system (2.1) to its reduced echelon form. Explain how you use the assumption on the coefficients during performing row operations.

(b) Explain whether the system (2.1) has a nontrivial solution.

(c) Discuss the existence and uniqueness of solutions to the following system. Justify your answer without solving the equations.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_1 + 2x_2 + 4x_3 + 8x_4 &= 21 \\ x_1 - x_2 + x_3 - x_4 &= 0 \\ x_1 - 3x_2 + 9x_3 - 27x_4 &= -74 \end{aligned}$$

# บทที่ 3

# Matrices

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถคำนวณและรู้จักสมบัติพื้นฐานของการคูณ การบวก การสลับเปลี่ยน (transponse) ของเมทริกซ์
- รู้จักนิยามและคุณสมบัติของเมทริกซ์มูลฐาน
- สามารถวิเคราะห์ได้ว่าเมทริกซ์ที่ให้มา มีอินเวอร์สหรือไม่ ถ้ามี จะหาอินเวอร์สได้อย่างไร
- สามารถหาอินเวอร์สและค่ากำหนดของเมทริกซ์จากการดำเนินการตามแถวนอนชั้นมูลฐานได้
- สามารถประยุกต์การหาอินเวอร์สของเมทริกซ์มาใช้ในการแก้สมการเชิงเส้นได้

# 3. Vectors and Matrices

- review on vectors
- matrix notation
- special matrices
- matrix operations
- inverse of matrices
- elementary matrices
- determinants
- linear equations in matrix form



## Vector notation

$n$-vector $x$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- also written as $x = (x_1, x_2, \dots, x_n)$
- set of $n$-vectors is denoted $\mathbf{R}^n$ (Euclidean space)
- $x_i$: $i$th **element** or **component** or **entry** of $x$
- $x$ is also called a column vector
- $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$ is called a row vector

unless stated otherwise, a vector typically means a column vector



## Special vectors

**zero vectors**: $x = (0, 0, \dots, 0)$

**all-ones vectors**: $x = (1, 1, \cdots, 1)$ (we will denote it by $\mathbf{1}$)

**standard unit vectors:** $e_k$ has only 1 at the $k$th entry and zero otherwise

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(standard unit vectors in $\mathbf{R}^3$)

**unit vectors**: any vector $u$ whose norm (magnitude) is $1$, *i.e.*,

$$\|u\| \triangleq \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = 1$$

example: $u = (1/\sqrt{2}, 2/\sqrt{6}, -1/\sqrt{2})$

## Vector operations

**scalar multiplication** of a vector $x$ with a scalar $\alpha$

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

**addition and subtraction** of two $n$-vector $x, y$

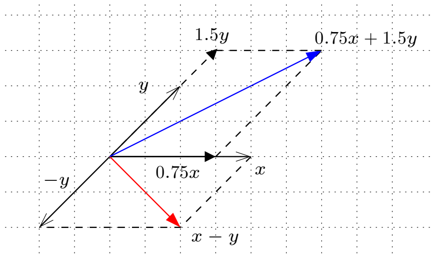
$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad x - y = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{bmatrix}$$



## Geometrical interpretation

for $n \le 3$: $x$ is a point with coordinates $x_i$

**example:** $x = (4, 0)$, $y = (2, 2)$




$$0.75x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad 1.5y = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad 0.75x + 1.5y = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad x - y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



## Inner products

**definition:** the inner product of two $n$-vectors $x, y$ is

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

also known as the **dot product** of vectors $x, y$

**notation:** $x^T y$

**properties** ✎

- $(\alpha x)^T y = \alpha(x^T y)$ for scalar $\alpha$
- $(x + y)^T z = x^T z + y^T z$
- $x^T y = y^T x$

## Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

**properties**

- also written $\|x\|_2$ to distinguish from other norms
- $\|\alpha x\| = |\alpha| \|x\|$ for scalar $\alpha$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
- $\|x\| \geq 0$ and $\|x\| = 0$ only if $x = 0$

**interpretation**

- $\|x\|$ measures the *magnitude* or length of $x$
- $\|x - y\|$ measures the *distance* between $x$ and $y$



## Matrix notation

an $m \times n$ matrix $A$ is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \text{or} \quad A = [a_{ij}]_{m \times n}$$

- $a_{ij}$ are the **elements**, or **coefficients**, or **entries** of $A$
- set of $m \times n$-matrices is denoted $\mathbf{R}^{m \times n}$
- $A$ has $m$ rows and $n$ columns ($m, n$ are the **dimensions**)
- the $(i, j)$ entry of $A$ is also commonly denoted by $A_{ij}$
- $A$ is called a **square** matrix if $m = n$



## Special matrices

**zero matrix:** $A = 0$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$a_{ij} = 0$, for $i = 1, \dots, m, j = 1, \dots, n$

**identity matrix:** $A = I$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

a square matrix with $a_{ii} = 1, a_{ij} = 0$ for $i \neq j$

**diagonal matrix:**

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{bmatrix}$$

a square matrix with $a_{ij} = 0$ for $i \neq j$

**triangular matrix:**

a square matrix with zero entries in a triangular part

**upper triangular**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$a_{ij} = 0$ for $i \geq j$

**lower triangular**

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$a_{ij} = 0$ for $i \leq j$



## Addition and scalar multiplication

**addition** of two $m \times n$-matrices $A$ and $B$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \ldots & a_{mn} + b_{mn} \end{bmatrix}$$

**scalar multiplication** of an $m \times n$-matrix $A$ with a scalar $\beta$

$$\beta A = \begin{bmatrix} \beta a_{11} & \beta a_{12} & \ldots & \beta a_{1n} \\ \beta a_{21} & \beta a_{22} & \ldots & \beta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta a_{m1} & \beta a_{m2} & \ldots & \beta a_{mn} \end{bmatrix}$$



## Multiplication

product of $m \times r$-matrix $A$ with $r \times n$-matrix $B$:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \sum_{k=1}^{r} a_{ik}\, b_{kj}$$

dimensions must be compatible: $\#$ of columns in $A = \#$ of rows in $B$

- $(AB)_{ij}$ is the dot product of the $i^{\text{th}}$ row of $A$ and the $j^{\text{th}}$ column of $B$
- $AB \neq BA$ in general ! (even if the dimensions make sense)
- there are exceptions, $e.g.$, $AI = IA$ for all square $A$
- $A(B + C) = AB + AC$

## Matrix transpose

the transpose of an $m \times n$-matrix $A$ is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

**properties** ✎

- $A^T$ is $n \times m$
- $(A^T)^T = A$
- $(\alpha A + B)^T = \alpha A^T + B^T, \quad \alpha \in \mathbf{R}$
- $(AB)^T = B^T A^T$
- a square matrix $A$ is called **symmetric** if $A = A^T$, *i.e.*, $a_{ij} = a_{ji}$



## Block matrix notation

**example:** $2 \times 2$-block matrix $A$

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

for example, if $B, C, D, E$ are defined as

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 7 \\ 1 & 9 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -4 & 1 & -1 \end{bmatrix}$$

then $A$ is the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 & 7 \\ 3 & 8 & 1 & 9 & 1 \\ 0 & 1 & -4 & 1 & -1 \end{bmatrix}$$

note: dimensions of the blocks must be compatible



## Column and Row partitions

write an $m \times n$-matrix $A$ in terms of its columns or its rows

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_m^T \end{bmatrix}$$

- $a_j$ for $j = 1, 2, \ldots, n$ are the columns of $A$
- $b_i^T$ for $i = 1, 2, \ldots, m$ are the rows of $A$

**example:**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 9 & 0 \end{bmatrix}$$

the column and row vectors are

$$a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_1^T = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \quad b_2^T = \begin{bmatrix} 4 & 9 & 0 \end{bmatrix}$$

## Matrix-vector product

product of $m \times n$-matrix $A$ with $n$-vector $x$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

- dimensions must be compatible: # columns in $A$ = # elements in $x$

if $A$ is partitioned as $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, then

$$Ax = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

- $Ax$ is a linear combination of the column vectors of $A$
- the coefficients are the entries of $x$



## Product with standard unit vectors

**post-multiply with a column vector**

$$Ae_k = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = \text{the } k\text{th column of } A$$

**pre-multiply with a row vector**

$$\begin{aligned} e_k^T A &= \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} = \text{the } k\text{th row of } A \end{aligned}$$



## Trace

**Definition:**

trace of a square matrix $A$ is the sum of the diagonal entries in $A$

$$\mathbf{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

**example:**

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

trace of $A$ is $2 - 1 + 6 = 7$

**properties** ✎

- $\mathbf{tr}(A^T) = \mathbf{tr}(A)$
- $\mathbf{tr}(\alpha A + B) = \alpha\,\mathbf{tr}(A) + \mathbf{tr}(B)$
- $\mathbf{tr}(AB) = \mathbf{tr}(BA)$

## Inverse of matrices

**Definition:**

a *square* matrix $A$ is called **invertible** or **nonsingular** if there exists $B$ s.t.

$$AB = BA = I$$

- $B$ is called an **inverse** of $A$
- it is also true that $B$ is invertible and $A$ is an inverse of $B$
- if no such $B$ can be found $A$ is said to be **singular**

assume $A$ is invertible

- an inverse of $A$ is unique
- the inverse of $A$ is denoted by $A^{-1}$



assume $A, B$ are invertible

**Facts** ✎

- $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$ for nonzero $\alpha$
- $A^T$ is also invertible and $(A^T)^{-1} = (A^{-1})^T$
- $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- $(A+B)^{-1} \neq A^{-1} + B^{-1}$



## Inverse of $2 \times 2$ matrices

the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if

$$ad - bc \neq 0$$

and its inverse is given by

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**example:**

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

## Elementary matrices

**Definition:** a matrix obtained by performing a *single* row operation on the identity matrix $I_n$ is called an **elementary** matrix

**examples:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \quad \text{add } k \text{ times the first row to the third row of } I_3$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \quad \text{multiply a nonzero } k \text{ with the second row of } I_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{interchange the second and the third rows of } I_3$$

an elementary matrix is often denoted by $E$



## Inverse operations

row operations on $E$ that produces $I$ and vice versa

| $I \to E$ | $E \to I$ |
|---|---|
| add $k$ times row $i$ to row $j$ | add $-k$ times row $i$ to row $j$ |
| multiply row $i$ by $k \neq 0$ | multiply row $i$ by $1/k$ |
| interchange row $i$ and $j$ | interchange row $i$ and $j$ |

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -k & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



**Facts**

- every elementary matrix is invertible
- the inverse is also an elementary matrix

from the examples in page 3-23

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -k & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

## Row operations by matrix multiplication

assume $A$ is $m \times n$ and $E$ is obtained by performing a row operation on $I_m$

$EA =$ the matrix obtained by performing this same row operation on $A$

**example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

- add $-2$ times the third row to the second row of $A$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad EA = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$



- multiply 2 with the first row of $A$

$$E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad EA = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

- interchange the first and the third rows of $A$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad EA = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$



## Inverse via row operations

assume $A$ is invertible

- $A$ is reduced to $I$ by a finite sequence of row operations

$$E_1, E_2, \ldots, E_k$$

  such that

$$E_k \cdots E_2 E_1 A = I$$

- the reduced echelon form of $A$ is $I$
- the inverse of $A$ is therefore given by the product of elementary matrices

$$A^{-1} = E_k \cdots E_2 E_1$$

**example:** write the augmented matrix $\left[A \mid I\right]$

$$\left.\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right.$$

and apply row operations until the left side is reduced to $I$

$$\begin{array}{r} -2R_2 + R_1 \to R_1 \\ -R_2 + R_3 \to R_3 \end{array} \quad \begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & -1 & 1 \end{array}$$

$$R_1 \leftrightarrow R_2 \quad \begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -1 & 1 \end{array}$$

$$-3R_2 + R_3 \to R_3 \quad \begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 0 & -3 & 5 & 1 \end{array}$$



$$R_3/(-2) \to R_3 \quad \begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \end{array}$$

$$R_2 \leftrightarrow R_3 \quad \begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}$$

$$-2R_2 + R_1 \to R_1 \quad \begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 6 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}$$

$$-R_3 + R_1 \to R_1 \quad \begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 8 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}$$

the inverse of $A$ is

$$\begin{bmatrix} -4 & 8 & 1 \\ \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\ 1 & -2 & 0 \end{bmatrix}$$



## Invertible matrices

☞ **Theorem:** for a square matrix $A$, the following statements are equivalent

1. $A$ is invertible
2. $Ax = 0$ has only the trivial solution ($x = 0$)
3. the reduced echelon form of $A$ is $I$
4. $A$ is expressible as a product of elementary matrices

## Inverse of special matrices

**diagonal matrix**

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{bmatrix}$$

a diagonal matrix is invertible iff the diagonal entries are all nonzero

$$a_{ii} \neq 0, \quad i = 1, 2, \ldots, n$$

the inverse of $A$ is given by

$$A^{-1} = \begin{bmatrix} 1/a_1 & 0 & \cdots & 0 \\ 0 & 1/a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1/a_n \end{bmatrix}$$

the diagonal entries in $A^{-1}$ are the inverse of the diagonal entries in $A$



**triangular matrix:**

**upper triangular**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$a_{ij} = 0$ for $i \geq j$

**lower triangular**

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$a_{ij} = 0$ for $i \leq j$

a triangular matrix is invertible iff the diagonal entries are all nonzero

$$a_{ii} \neq 0, \quad \forall i = 1, 2, \ldots, n$$

more is true ...

- product of lower (upper) triangular matrices is lower (upper) triangular
- the inverse of a lower (upper) triangular matrix is lower (upper) triangular



**symmetric matrix:** $A = A^T$

- for any square matrix $A$, $AA^T$ and $A^TA$ are always symmetric
- if $A$ is symmetric and invertible, then $A^{-1}$ is symmetric
- if $A$ is invertible, then $AA^T$ and $A^TA$ are also invertible

## Determinants

the determinant is a *scalar value* associated with a square matrix $A$

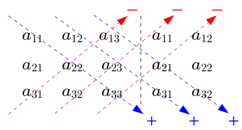
commonly denoted by $\det(A)$ or $|A|$

determinants of $2 \times 2$ matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

determinants of $3 \times 3$ matrices: let $A = \{a_{ij}\}$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$






for a square matrix of any order, it can be computed by

- cofactor expansion
- performing elementray row operations



## Minor and Cofactor

**Minor of entry $a_{ij}$:** denoted by $M_{ij}$

- the determinant of the resulting submatrix after deleting the $i$th row and $j$th column of $A$

**Cofactor of entry $a_{ij}$:** denoted by $C_{ij}$

- $C_{ij} = (-1)^{(i+j)} M_{ij}$

**example:**

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad M_{23} = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4, \quad C_{23} = (-1)^{(2+3)} M_{23} = 4$$

## Determinants by Cofactor Expansion

**Theorem:** the determinant of an $n \times n$-matrix $A$ is given by

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

regardless of which row or column of $A$ is chosen

**example:** pick the first row to compute $\det(A)$

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{aligned} \det(A) &= 3(-1)^2 \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} - 2(-1)^4 \begin{vmatrix} 5 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 3(1)(2) + (-1)(8) - 2(1)(-5) = 8 \end{aligned}$$



## Basic properties of determinants

☞ let $A, B$ be any square matrices

- $\det(A) = \det(A^T)$
- if $A$ has a row of zeros or a column of zeros, then $\det(A) = 0$
- $\det(A + B) \neq \det(A) + \det(B)$ !

determinants of special matrices:

- the determinant of a diagonal or triangular matrix is given by the product of the diagonal entries
- $\det(I) = 1$

(these properties can be proved from the def. of cofactor expansion)



☞ another basic properties; suppose the following is true

- $A$ and $B$ are equal except for the entries in their $k$th row (column)
- $C$ is defined as that matrix identical to $A$ and $B$ except that its $k$th row (column) is the sum of the $k$th rows (columns) of $A$ and $B$

then we have

$$\det(C) = \det(A) + \det(B)$$

example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 0, \quad \det(B) = -1, \quad \det(C) = -1$$

## Determinants under row operations

- multiply $k$ to a row or a column

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- interchange between two rows or two columns

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- add $k$ times the $i$th row (column) to the $j$th row (column)

$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



(the proof of determinants under row operations is left as an exercise )

example: $B$ is obtained by performing the following operations on $A$

$$R_2 + 3R_1 \to R_2, \quad R_3 \leftrightarrow R_1, \quad -4R_1 \to R_1$$

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 1 & 0 \\ -3 & -3 & 3 \end{bmatrix} \quad \Longrightarrow \quad \det(B) = (-4)\cdot(-1)\cdot 1 \cdot \det(A)$$

the changes of det. under elementary operations lead to obvious facts ✎

- $\det(\alpha A) = \alpha^n \det(A), \quad \alpha \neq 0$
- If $A$ has two rows (columns) that are equal, then $\det(A) = 0$



## Determinants of elementary matrices

let $B$ be obtained by performing a row operation on $A$ then

$$B = EA \quad \text{and} \quad \det(B) = \det(EA)$$

$$E = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = k\det(A) \quad (\det(E) = k)$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = -\det(A) \quad (\det(E) = -1)$$

$$E = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = \det(A) \quad (\det(E) = 1)$$

**conclusion:** $\det(EA) = \det(E)\det(A)$

## Determinants of product and inverse

☞ let $A, B$ be $n \times n$ matrices

- $A$ is invertible if and only if $\det(A) \neq 0$
- if $A$ is invertible, then $\det(A^{-1}) = 1/\det(A)$
- $\det(AB) = \det(A)\det(B)$



## Adjoint

the adjoint of $A$ is the transpose of the matrix of cofactors from $A$

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

if $A$ is invertible then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

**Proof.**

- the cofactor expansion using the cofactors from different row is zero

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \ldots + a_{in}C_{kn} = 0, \quad \text{for } i \neq k$$

- $A \, \text{adj}(A) = \det(A) \cdot I$



## Linear equation in matrix form

the linear system of $m$ equations in $n$ variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

in matrix form: $Ax = b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

## Applications

a set of linear equations $Ax = b$ (with $A$ $m \times n$) is

- **square** if $m = n$ ($A$ is square)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **underdetermined** if $m < n$ ($A$ is fat)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **overdetermined** if $m > n$ ($A$ is skinny)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



## Cramer's rule

consider a linear system $Ax = b$ when $A$ is **square**

if $A$ is invertible then the solution is unique and given by

$$x = A^{-1}b$$

each component of $x$ can be calculated by using the Cramer's rule

**Cramer's rule**

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \ldots, \quad x_n = \frac{|A_n|}{|A|}$$

where $A_j$ is the matrix obtained by replacing $b$ in the $j$th column of $A$

(its proof is left as an exercise)



**example:**

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

since $\det(A) = 8$, $A$ is invertible and the solution is

$$x = A^{-1}b = \frac{1}{8} \begin{bmatrix} 2 & 0 & 2 \\ -8 & 8 & -16 \\ -5 & 4 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}$$

using Cramer's rule gives

$$x_1 = \frac{1}{8} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \\ 2 & -1 & 2 \end{vmatrix}, \quad x_2 = \frac{1}{8} \begin{vmatrix} 3 & 2 & -2 \\ 5 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix}, \quad x_3 = \frac{1}{8} \begin{vmatrix} 3 & 1 & 2 \\ 5 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

which yields

$$x_1 = 1, \quad x_2 = -5, \quad x_3 = -2$$

## MATLAB commands

some commonly used commands for working with matrices

- `eye(n)` produces an identity matrix of size $n$
- `zeros(m,n)` creates a zero matrix of size $m \times n$
- `inv(A)` finds the inverse of $A$
- `det(A)` finds the determinant of $A$
- `trace(A)` finds the trace of $A$

to solve $Ax = b$ when $A$ is square use

- `A\b` (compute $A^{-1}b$)



## References

Chapter 1-2 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

Lecture note on

*Matrices and Vectors*, EE103, L. Vandenberghe, UCLA

## Exercises

1. Let $A$ be a square matrix of size $n \times n$. Prove the following statements.

   (a) If $B$ is the matrix that results when a single row or single columns of $A$ is multiplied by a scalar $k$, then $\det(B) = k\det(B)$.

   (b) If $B$ is the matrix that results when two rows or two columns of $A$ are interchanged, then $\det(B) = -\det(A)$.

   (c) If $B$ is the matrix that results when a multiple of one row of $A$ is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

   Your proof must be valid for any $A$.

2. If $A$ and $B$ are square matrices of the same size, then prove that

$$\det(AB) = \det(A)\det(B).$$

3. Let $v_1, v_2, \ldots, v_n$ be vectors in $R^n$. Prove that the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

   has only the solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ ($\alpha_k$'s are scalar) if and only if the determinant of

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

   is NOT zero.

4. A square matrix is called skew-symmetric if $A^T = -A$.

   (a) Give examples of $3 \times 3$ and $2 \times 2$ skew-symmetric matrices.

   (b) If $A$ is skew-symmetric, prove that $A^{-1}$ is skew-symmetric.

   (c) If $A$ and $B$ are skew-symmetric matrices, then so are $A^T, A + B, A - B$, and $kA$ for any scalar $k$.

   (d) If $A$ is skew-symmetric, what is $\det(A)$ ? Verify your result with the examples in part a).

5. Without directly evaluating the determinant, show that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0.$$

# บทที่ 4

# Vector spaces

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- รู้จักนิยามของปริภูมิเวกเตอร์ (vector space) และสามารถวิเคราะห์ได้ว่าเซตหนึ่งๆ ที่ให้มา เป็นปริภูมิเวกเตอร์หรือไม่
- สามารถวิเคราะห์ได้ว่าเวคเตอร์กลุ่มหนึ่งที่ให้มา มีความเป็นอิสระเชิงเส้น (linear independence) หรือไม่
- สามารถหาฐานหลักและมิติ (basis and dimension) ของปริภูมิเวกเตอร์หนึ่งๆ ได้
- สามารถหาพิกัด (coordinate) ของเวคเตอร์เทียบกับฐานหลักหนึ่งๆ ได้
- สามารถหาปริภูมิสู่ศูนย์ (nullspace) และปริภูมิพิสัย (range space) ของเมทริกซ์หนึ่งๆ ได้

EE202 - EE MATH II Jitkomut Songsiri

# 4. Vector spaces

- definition
- linear independence
- basis and dimension
- coordinate and change of basis
- range space and null space
- rank and nullity



## Vector space

a vector space or linear space (over **R**) consists of

- a set $\mathcal{V}$
- a vector sum $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
- a scalar multiplication : $\mathbf{R} \times \mathcal{V} \to \mathcal{V}$
- a distinguished element $0 \in \mathcal{V}$

which satisfy a list of properties



- $x + y \in \mathcal{V} \quad \forall x, y \in \mathcal{V}$ (closed under addition)
- $x + y = y + x$, $\forall x, y \in \mathcal{V}$ ($+$ is commutative)
- $(x + y) + z = x + (y + z)$, $\forall x, y, z \in \mathcal{V}$ ($+$ is associative)
- $0 + x = x$, $\forall x \in \mathcal{V}$ ($0$ is additive identity)
- $\forall x \in \mathcal{V}\ \exists(-x) \in \mathcal{V}$ s.t. $x + (-x) = 0$ (existence of additive inverse)
- $\alpha x \in \mathcal{V}$ for any $\alpha \in \mathbf{R}$ (closed under scalar multiplication)
- $(\alpha\beta)x = \alpha(\beta x)$, $\forall \alpha, \beta \in \mathbf{R}\ \forall x \in \mathcal{V}$ (scalar multiplication is associative)
- $\alpha(x + y) = \alpha x + \alpha y$, $\forall \alpha \in \mathbf{R}\ \forall x, y \in \mathcal{V}$ (right distributive rule)
- $(\alpha + \beta)x = \alpha x + \alpha y$, $\forall \alpha, \beta \in \mathbf{R}\ \ \forall x \in \mathcal{V}$ (left distributive rule)
- $\mathbf{1}x = x$, $\forall x \in \mathcal{V}$ ($1$ is multiplicative identity)

**notation**

- $(\mathcal{V}, \mathbf{R})$ denotes a vector space $\mathcal{V}$ over **R**
- an element in $\mathcal{V}$ is called a **vector**

**Theorem:** let $u$ be a vector in $\mathcal{V}$ and $k$ a scalar; then

- $0u = 0$ (multiplication with zero gives the zero vector)
- $k0 = 0$ (multiplication with the zero vector gives the zero vector)
- $(-1)u = -u$ (multiplication with $-1$ gives the additive inverse)
- if $ku = 0$, then $k = 0$ or $u = 0$



roughly speaking, a vector space must satisfy the following operations

1. **vector addition**

$$x, y \in \mathcal{V} \quad \Rightarrow \quad x + y \in \mathcal{V}$$

2. **scalar multiplication**

$$\text{for any } \alpha \in \mathbf{R}, \; x \in \mathcal{V} \quad \Rightarrow \quad \alpha x \in \mathcal{V}$$

the second condition implies that a vector space contains the **zero vector**

$$0 \in \mathcal{V}$$

in otherwords, if $\mathcal{V}$ is a vector space then $0 \in \mathcal{V}$

(but the converse is *not true*)



**examples:** the following sets are vector spaces (over **R**)

- $\mathbf{R}^n$
- $\{0\}$
- $\mathbf{R}^{m \times n}$
- $\mathbf{C}^{m \times n}$: set of $m \times n$-complex matrices
- $\mathbf{P}_n$: set of polynomials of degree $\leq n$

$$\mathbf{P}_n = \{p(t) \mid p(t) = a_0 + a_1 t + \cdots + a_n t^n\}$$

- $\mathbf{S}^n$: set of symmetric matrices of size $n$
- $C(-\infty, \infty)$: set of real-valued continuous functions on $(-\infty, \infty)$
- $C^n(-\infty, \infty)$: set of real-valued functions with continuous $n$th derivatives on $(-\infty, \infty)$

check whether any of the following sets is a vector space (over **R**)

- $\{0, 1, 2, 3, \ldots\}$
- $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- $\left\{ x \in \mathbf{R}^2 \mid x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad x_1 \in \mathbf{R} \right\}$
- $\{p(x) \in \mathbf{P}_2 \mid p(x) = a_1 x + a_2 x^2 \quad \text{for some } a_1, a_2 \in \mathbf{R}\}$



## Subspace

- a **subspace** of a vector space is a *subset* of a vector space which is itself a vector space
- a subspace is closed under vector addition and scalar multiplication

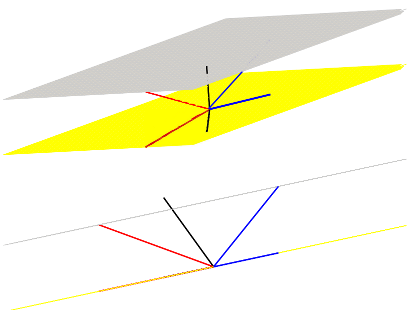
**examples:**

- $\{0\}$ is a subspace of $\mathbf{R}^n$
- $\mathbf{R}^{m \times n}$ is a subspace of $\mathbf{C}^{m \times n}$
- $\{x \in \mathbf{R}^2 \mid x_1 = 0\}$ is a subspace of $\mathbf{R}^2$
- $\{x \in \mathbf{R}^2 \mid x_2 = 1\}$ is not a subspace of $\mathbf{R}^2$
- $\left\{ \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is not a subspace of $\mathbf{R}^{2 \times 2}$
- the solution set $\{x \in \mathbf{R}^n \mid Ax = b\}$ for $b \neq 0$ is a not subspace of $\mathbf{R}^n$



**examples:** two hyperplanes; one is a subspace but the other one is not

$$2x_1 - 3x_2 + x_3 = 0 \quad \text{(yellow)}, \quad 2x_1 - 3x_2 + x_3 = 20 \quad \text{(grey)}$$



**black** = **red** + **blue**

$x = (-3, -2, 0)$ & $y = (1, -1, -5)$ are on the yellow plane, and so is $x + y$

$x = (-3, -2, 20)$ & $y = (1, -1, 15)$ are on the grey plane, but $x + y$ is not

## Linear Independence

**Definition:** a set of vectors $\{v_1, v_2, \dots, v_n\}$ is **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ are uniquely determined, i.e.,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

  implies $\alpha_k = \beta_k$ for $k = 1, 2, \dots, n$

- no vector $v_i$ can be expressed as a linear combination of the other vectors



**examples:**

- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}$ are independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ are independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}$ are not independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ are not independent



## Linear span

**Definition:** the linear span of a set of vectors

$$\{v_1, v_2, \dots, v_n\}$$

is the set of all linear combinations of $v_1, \dots, v_n$

$$\text{span}\{v_1, v_2, \dots, v_n\} = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, \dots, a_n \in \mathbf{R}\}$$

**example:**

$\text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is the set of $2 \times 2$ symmetric matrices

**Fact:** if $v_1, \dots, v_n$ are vectors in $\mathcal{V}$, $\text{span}\{v_1, \dots, v_n\}$ is a subspace of $\mathcal{V}$

## Basis and dimension

**Definition:** set of vectors $\{v_1, v_2, \cdots, v_n\}$ is a **basis** for a vector space $\mathcal{V}$ if

- $\{v_1, v_2, \ldots, v_n\}$ is linearly independent
- $\mathcal{V} = \text{span}\,\{v_1, v_2, \ldots, v_n\}$

equivalent condition: *every* $v \in \mathcal{V}$ can be *uniquely* expressed as

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

**Definition:** the **dimension** of $\mathcal{V}$, denoted $\dim(\mathcal{V})$, is the number of vectors in a basis for $\mathcal{V}$

**Theorem:** the number of vectors in *any* basis for $\mathcal{V}$ is the same

(we assign $\dim\{0\} = 0$ )



**examples:**

- $\{e_1, e_2, e_3\}$ is a standard basis for $\mathbf{R}^3$ $(\dim \mathbf{R}^3 = 3)$
- $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^2$ $(\dim \mathbf{R}^2 = 2)$
- $\{1, t, t^2\}$ is a basis for $\mathbf{P}_2$ $(\dim \mathbf{P}_2 = 3)$
- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^{2\times 2}$ $(\dim \mathbf{R}^{2\times 2} = 4)$
- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ cannot be a basis for $\mathbf{R}^3$ why ?
- $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ cannot be a basis for $\mathbf{R}^2$ why ?



## Coordinates

let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space $\mathcal{V}$

suppose a vector $v \in \mathcal{V}$ can be written as

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

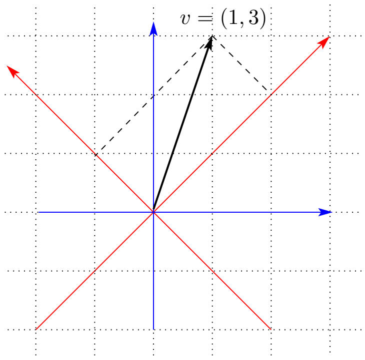
**Definition:** the coordinate vector of $v$ relative to the basis $S$ is

$$[v]_S = (a_1, a_2, \ldots, a_n)$$

- linear independence of vectors in $S$ ensures that $a_k$'s are *uniquely* determined by $S$ and $v$
- changing the basis yields a different coordinate vector

## Geometrical interpretation

new coordinate in a new reference axis




$$v = \begin{bmatrix}1\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$$



**examples:**

- $S = \{e_1, e_2, e_3\}$, $v = (-2, 4, 1)$

$$v = -2e_1 + 4e_2 + 1e_3, \quad [v]_S = (-2, 4, 1)$$

- $S = \{(-1, 2, 0), (3, 0, 0), (-2, 1, 1)\}$, $v = (-2, 4, 1)$

$$v = \begin{bmatrix}-2\\4\\1\end{bmatrix} = \frac{3}{2}\begin{bmatrix}-1\\2\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}3\\0\\0\end{bmatrix} + 1\begin{bmatrix}-2\\1\\1\end{bmatrix}, \quad [v]_S = (3/2, 1/2, 1)$$

- $S = \{1, t, t^2\}$, $v(t) = -3 + 2t + 4t^2$

$$v(t) = -3 \cdot 1 + 2 \cdot t + 4 \cdot t^2, \quad [v]_S = (-3, 2, 4)$$

- $S = \{1, t - 1, t^2 + t\}$, $v(t) = -3 + 2t + 4t^2$

$$v(t) = -5 \cdot 1 - 2 \cdot (t - 1) + 4 \cdot (t^2 + t), \quad [v]_S = (-5, -2, 4)$$



## Change of basis

let $U = \{u_1, \dots, u_n\}$ and $W = \{w_1, \dots, w_n\}$ be bases for a vector space $\mathcal{V}$

a vector $v \in \mathcal{V}$ has the coordinates relative to these bases as

$$[v]_U = (a_1, a_2, \dots, a_n), \quad [v]_W = (b_1, b_2, \dots, b_n)$$

suppose the coordinate vectors of $w_k$ relative to $U$ is

$$[w_k]_U = (c_{1k}, c_{2k}, \dots, c_{nk})$$

or in the matrix form as

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

the coordinate vectors of $v$ relative to $U$ and $W$ are related by

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \triangleq P \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- we obtain $[v]_U$ by multiplying $[v]_W$ with $P$
- $P$ is called the **transition** matrix from $W$ to $U$
- the columns of $P$ are the coordinate vectors of the basis vectors in $W$ relative to $U$

**Theorem** ✌

$P$ is invertible and $P^{-1}$ is the transition matrix from $U$ to $W$



**example:** find $[v]_U$, given

$$U = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \quad W = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad [v]_W = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

first, find the coordinate vectors of the basis vectors in $W$ relative to $U$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

from which we obtain the transition matrix

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$$

and $[v]_U$ is given by

$$[v]_U = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

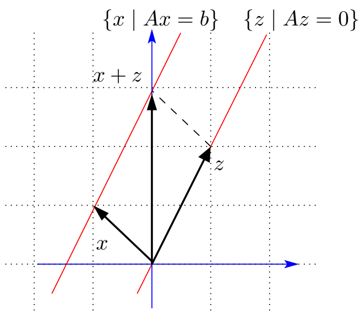


## Nullspace

the **nullspace** of an $m \times n$ matrix is defined as

$$\mathcal{N}(A) = \{x \in \mathbf{R}^n \mid Ax = 0\}$$

- the set of all vectors that are mapped to zero by $f(x) = Ax$
- the set of all vectors that are orthogonal to the rows of $A$
- if $Ax = b$ then $A(x + z) = b$ for all $z \in \mathcal{N}(A)$
- also known as **kernel** of $A$
- $\mathcal{N}(A)$ is a subspace of $\mathbf{R}^n$ ✎




$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \\ -6 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ 6 \\ 9 \end{bmatrix}$$

- $\mathcal{N}(A) = \{x \mid 2x_1 - x_2 = 0\}$
- the solution set of $Ax = b$ is $\{x \mid 2x_1 - x_2 = -3\}$
- the solution set of $Ax = b$ is the translation of $\mathcal{N}(A)$



## Zero nullspace matrix

- $A$ has a zero nullspace if $\mathcal{N}(A) = \{0\}$
- if $A$ has a zero nullspace and $Ax = b$ is solvable, the solution is unique
- columns of $A$ are independent

✌ **equivalent conditions:** $A \in \mathbf{R}^{n \times n}$

- $A$ has a zero nullspace
- $A$ is invertible or nonsingular
- columns of $A$ are a basis for $\mathbf{R}^n$



## Range space

the **range** of an $m \times n$ matrix $A$ is defined as

$$\mathcal{R}(A) = \{y \in \mathbf{R}^m \mid y = Ax \text{ for some } x \in \mathbf{R}^n \}$$

- the set of all $m$-vectors that can be expressed as $Ax$
- the set of all linear combinations of the columns of $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

$$\mathcal{R}(A) = \{y \mid y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n, \quad x \in \mathbf{R}^n\}$$

- the set of all vectors $b$ for which $Ax = b$ is solvable
- also known as the **column space** of $A$
- $\mathcal{R}(A)$ is a subspace of $\mathbf{R}^m$

## Full range matrices

$A$ has a full range if $\mathcal{R}(A) = \mathbf{R}^m$

☞ **equivalent conditions:**

- $A$ has a full range
- columns of $A$ span $\mathbf{R}^m$
- $Ax = b$ is solvable for *every* $b$
- $\mathcal{N}(A^T) = \{0\}$



## Bases for $\mathcal{R}(A)$ and $\mathcal{N}(A)$

$A$ and $B$ are row equivalent matrices, $i.e.$,

$$B = E_k \cdots E_2 E_1 A$$

**Facts** ☞

- elementary row operations *do not alter* $\mathcal{N}(A)$

$$\mathcal{N}(B) = \mathcal{N}(A)$$

- columns of $B$ are independent if and only if columns of $A$ are
- a given set of column vectors of $A$ forms a basis for $\mathcal{R}(A)$ if and only if the corresponding column vectors of $B$ form a basis for $\mathcal{R}(B)$



**example:** given a matrix $A$ and its row echelon form $B$:

$$A = \begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 6 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**basis for** $\mathcal{N}(A)$: from $\{x \mid Ax = 0\} = \{x \mid Bx = 0\}$, we read

$$x_1 + x_4 = 0, \quad x_2 + 2x_3 + x_4 = 0$$

define $x_3$ and $x_4$ as free variables, any $x \in \mathcal{N}(A)$ can be written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a linear combination of $(0, -2, 1, 0)$ and $(-1, -1, 0, 1)$

hence, a basis for $\mathcal{N}(A)$ is $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\dim \mathcal{N}(A) = 2$

**basis for** $\mathcal{R}(A)$: pick a set of the independent column vectors in $B$ (here pick the 1st and the 2nd columns)

the corresponding columns in $A$ form a basis for $\mathcal{R}(A)$:

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$\dim \mathcal{R}(A) = 2$



☞ **conclusion:** if $R$ is the row reduced echelon form of $A$

- the pivot column vectors of $R$ form a basis for the range space of $R$
- the column vectors of $A$ *corresponding* to the pivot columns of $R$ form a basis for the range space of $A$
- $\dim \mathcal{R}(A)$ is the number of leading 1's in $R$
- $\dim \mathcal{N}(A)$ is the number of free variables in solving $Rx = 0$



## Rank and Nullity

**rank** of a matrix $A \in \mathbf{R}^{m \times n}$ is defined as

$$\mathbf{rank}(A) = \dim \mathcal{R}(A)$$

**nullity** of a matrix $A \in \mathbf{R}^{m \times n}$ is

$$\mathbf{nullity}(A) = \dim \mathcal{N}(A)$$

**Facts** ☞

- $\mathbf{rank}(A)$ is maximum number of independent columns (or rows) of $A$

$$\mathbf{rank}(A) \leq \min(m, n)$$

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$

## Full rank matrices

for $A \in \mathbf{R}^{m \times n}$ we always have $\mathbf{rank}(A) \leq \min(m, n)$

we say $A$ is **full rank** if $\mathbf{rank}(A) = \min(m, n)$

- for **square** matrices, full rank means nonsingular (invertible)
- for **skinny** matrices $(m \geq n)$, full rank means columns are independent
- for **fat** matrices $(m \leq n)$, full rank means rows are independent



## Rank-Nullity Theorem

for any $A \in \mathbf{R}^{m \times n}$,

$$\mathbf{rank}(A) + \dim \mathcal{N}(A) = n$$

Proof:

- a homogeneous linear system $Ax = 0$ has $n$ variables
- these variables fall into two categories
  - leading variables
  - free variables
- $\#$ of leading variables $= \#$ of leading 1's in reduced echelon form of $A$ $= \mathbf{rank}(A)$
- $\#$ of free variables $=$ nullity of $A$



## MATLAB commands

- `rref(A)` produces the reduced row echelon form of $A$

```
>> A = [-1 2 4 1;0 1 2 1;2 3 6 5]
A =
    -1     2     4     1
     0     1     2     1
     2     3     6     5

>> rref(A)

ans =

     1     0     0     1
     0     1     2     1
     0     0     0     0
```

- `rank(A)` provides an estimate of the rank of $A$

- `null(A)` gives normalized vectors in a basis for $\mathcal{N}(A)$

```
>> A

A =

     1    -3     2
     2    -6     4
     3    -9     6

>> U = null(A)

U =

   -0.8729   -0.4082
   -0.4364    0.4082
   -0.2182    0.8165
```

(and we can verify that $AU = 0$)



## References

Chapter 4 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

Lecture note on

*Linear algebra review*, EE263, S. Boyd, Stanford University

Lecture note on

*Theory of linear equations*, EE103, L. Vandenberghe, UCLA

## Exercises

1. Let $x$ and $y$ be any two vectors in $\mathbf{R}^n$. We say $x$ and $y$ are orthogonal if $x^T y = 0$, i.e.,

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = 0.$$

   Define $S$ a set of all vectors in $\mathbf{R}^n$ that are orthogonal to the hyperplane

$$H = \{x \in \mathbf{R}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b\}.$$

   In other words, if $x \in S$ then $x^T y = 0$ for all $y \in H$. Show that $S$ is a subspace of $\mathbf{R}^n$. Find a basis for $S$ and its dimension.

2. Let

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 5 & 7 & -8 \\ 4 & 2 & -10 \\ 1 & 3 & 0 \end{bmatrix}$$

   and define $\mathcal{V}$ as a set of vectors $b$ for which $Ax = b$ is solvable.

   (a) Show that $\mathcal{V}$ is a vector space. Find a basis for $\mathcal{V}$ and determine its dimension.

   (b) What is the rank and nullity of $A$ ?

   (c) Find a basis for the row space of $A$.

3. Determine whether each of the following sets is a subspace. If one is, find a basis and its dimension.

   (a) $S = \{(1,2), (3,1), (0,0)\}$.

   (b) $S = \{x \in \mathbf{R}^n \mid x_1 + x_n = 0\}$.

   (c) $S = \{A \in \mathbf{R}^{3\times 3} \mid A^T = -A\}$.

   (d) Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ ($b$ is nonzero). Let $S$ be the set of all vectors $y \in \mathbf{R}^m$ obtained by

$$y = Ax + b$$

   for any $x \in \mathbf{R}^n$.

   (e) $S = \{p \in \mathbf{P}_n \mid p(x) = a_0 + a_1 x + \cdots + a_n x^n \text{ with } a_1 + a_2 + \cdots + a_n = 0\}$.

   (f) $S = \{p \in \mathbf{P}_n \mid p(x) = a_0 + a_1 x + \cdots + a_n x^n \text{ with } a_1 = 1\}$.

   (g) $S = \{x \in \mathbf{R}^n \mid a^T x \geq 0\}$, for some fixed $a \in \mathbf{R}^n$.

   (h) $S = \{A \in \mathbf{R}^{n\times n} \mid A^2 = A\}$.

   (i) $S = \{A \in \mathbf{R}^{n\times n} \mid a_{11} + a_{22} + \cdots + a_{nn} = 0\}$.

   (j) $S = \{p \in \mathbf{P}_n \mid \text{all the roots of } p(x) \text{ are } 0\}$.

   (k) $S = \{A \in \mathbf{S}^n \mid \text{all the eigenvalues of } A \text{ are nonnegative}\}$.

   (l) $S = \operatorname{span}\{(1,2,0,1), (0,0,1,3), (-4,-8,1,-1), (2,4,-3,-7), (1,2,-1,-2)\}$.

# บทที่ 5

# Linear transformations

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถวิเคราะห์และจำแนกได้ว่าการแปลงหนึ่งๆ ที่ให้มา เป็นการแปลงเชิงเส้นหรือไม่
- สามารถหาแก่นกลาง (kernel) และพิสัย (range) ของการแปลงเชิงเส้นได้
- สามารถวิเคราะห์ได้ว่าการแปลงเชิงเส้นหนึ่งๆ เป็นการแปลงแบบหนึ่งต่อหนึ่ง (one-to-one) หรือแบบทั่วถึง (onto) หรือไม่
- สามารถวิเคราะห์ได้ว่าการแปลงเชิงเส้นหนึ่งๆ มีการแปลงผกผัน (inverse) หรือไม่ หากมี จะหาได้อย่างไร

# 5. Linear Transformation

- linear transformation
- matrix transformation
- kernel and range
- isomorphism
- composition
- inverse transformation



## Transformation

let $X$ and $Y$ be vector spaces

a **transformation** $T$ from $X$ to $Y$, denoted by

$$T : X \to Y$$

is an assignment taking $x \in X$ to $y = T(x) \in Y$,

$$T : X \to Y, \quad y = T(x)$$

- **domain** of $T$, denoted $\mathcal{D}(T)$ is the collection of all $x \in X$ for which $T$ is defined
- vector $T(x)$ is called the **image** of $x$ under $T$
- collection of all $y = T(x) \in Y$ is called the **range** of $T$, denoted $\mathcal{R}(T)$



**example 1** define $T : \mathbf{R}^3 \to \mathbf{R}^2$ as

$$\begin{aligned} y_1 &= -x_1 + 2x_2 + 4x_3 \\ y_2 &= -x_2 + 9x_3 \end{aligned}$$

where $x \in \mathbf{R}^3$ and $y \in \mathbf{R}^2$

**example 2** define $T : \mathbf{R}^3 \to \mathbf{R}$ as

$$y = \sin(x_1) + x_2 x_3 - x_3^2$$

where $x \in \mathbf{R}^3$ and $y \in \mathbf{R}$

**example 3** general transformation $T : \mathbf{R}^n \to \mathbf{R}^m$

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_n) \\ y_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ y_m &= f_m(x_1, x_2, \ldots, x_n) \end{aligned}$$

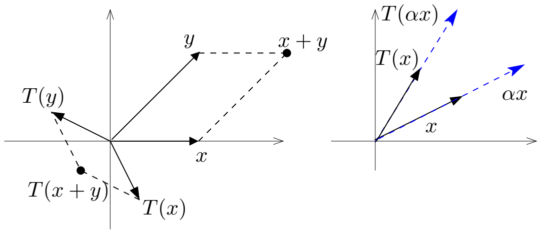
where $f_1, f_2, \ldots, f_m$ are real-valued functions of $n$ variables

## Linear transformation

let $X$ and $Y$ be vector spaces over **R**

**Definition:** a transformation $T : X \to Y$ is **linear** if

- $T(x+z) = T(x) + T(z), \quad \forall x, y \in X$ (additivity)
- $T(\alpha x) = \alpha T(x), \quad \forall x \in X, \forall \alpha \in \mathbf{R}$ (homogeneity)






## Examples

✎ which of the following is a linear transformation ?

- **matrix transformation** $T : \mathbf{R}^n \to \mathbf{R}^m$

$$T(x) = Ax, \quad A \in \mathbf{R}^{m \times n}$$

- **affine transformation** $T : \mathbf{R}^n \to \mathbf{R}^m$

$$T(x) = Ax + b, \quad A \in \mathbf{R}^{m \times n}, \quad b \in \mathbf{R}^m$$

- $T : \mathbf{P}_n \to \mathbf{P}_{n+1}$

$$T(p(t)) = tp(t)$$

- $T : \mathbf{P}_n \to \mathbf{P}_n$

$$T(p(t)) = p(t+1)$$



- $T : \mathbf{R}^{m \times n} \to \mathbf{R}^{n \times m}, \quad T(X) = X^T$
- $T : \mathbf{R}^{n \times n} \to \mathbf{R}, \quad T(X) = \det(X)$
- $T : \mathbf{R}^{n \times n} \to \mathbf{R}, \quad T(X) = \mathbf{tr}(X)$
- $T : \mathbf{R}^n \to \mathbf{R}, \quad T(x) = \|x\| \triangleq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
- $T : \mathbf{R}^n \to \mathbf{R}^n, \quad T(x) = 0$

denote $F(-\infty, \infty)$ the set of all real-valued functions on $(-\infty, \infty)$

- $T : C^1(-\infty, \infty) \to F(-\infty, \infty)$

$$T(f) = f'$$

- $T : C(-\infty, \infty) \to C^1(-\infty, \infty)$

$$T(f) = \int_0^t f(s)ds$$

## Examples of matrix transformation

$T: \mathbf{R}^n \to \mathbf{R}^m$

$$T(x) = Ax, \quad A \in \mathbf{R}^{m\times n}$$

**zero transformation:** $T: \mathbf{R}^n \to \mathbf{R}^m$

$$T(x) = 0 \cdot x = 0$$

$T$ maps every vector into the zero vector

**identity operator:** $T: \mathbf{R}^n \to \mathbf{R}^n$

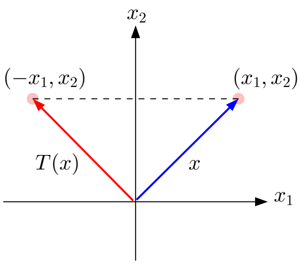
$$T(x) = I_n \cdot x = x$$

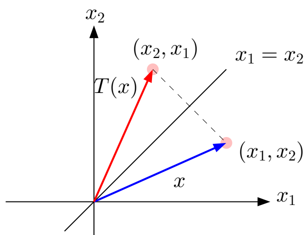
$T$ maps a vector into itself



**reflection operator:** $T: \mathbf{R}^n \to \mathbf{R}^n$

$T$ maps each point into its symmetric image about an axis or a line



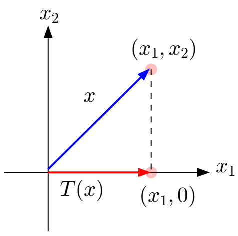



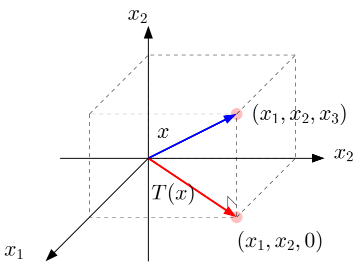

$$T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x \qquad T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$



**projection operator:** $T: \mathbf{R}^n \to \mathbf{R}^n$

$T$ maps each point into its orthogonal projection on a line or a plane

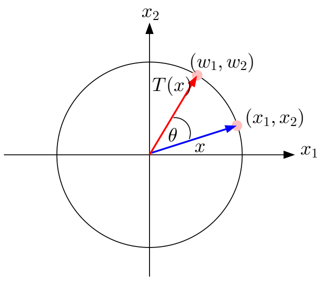



$$T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \qquad T(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

**rotation operator:** $T : \mathbf{R}^n \to \mathbf{R}^n$

$T$ maps points along circular arcs




$T$ rotates $x$ through an angle $\theta$

$$w = T(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$$



## Image of linear transformation

let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and a basis for $\mathcal{V}$ is

$$S = \{v_1, v_2, \dots, v_n\}$$

let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation

the image of any vector $v \in \mathcal{V}$ under $T$ can be expressed by

$$T(v) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$$

where $a_1, a_2, \dots, a_n$ are coefficients used to express $v$, *i.e.*,

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

(follow from the linear property of $T$)



## Kernel and Range

let $T : X \to Y$ be a linear transformation from $X$ to $Y$

**Definitions:**

**kernel** of $T$ is the set of vectors in $X$ that $T$ maps into $0$

$$\mathbf{ker}(T) = \{x \in X \mid T(x) = 0\}$$

**range** of $T$ is the set of all vectors in $Y$ that are images under $T$

$$\mathcal{R}(T) = \{y \in Y \mid y = T(x), \quad x \in X\}$$

**Theorem** ✎

- $\mathbf{ker}(T)$ is a subspace of $X$
- $\mathcal{R}(T)$ is a subspace of $Y$

**matrix transformation:** $T : \mathbf{R}^n \to \mathbf{R}^m, \quad T(x) = Ax$

- $\mathbf{ker}(T) = \mathcal{N}(A)$: kernel of $T$ is the nullspace of $A$
- $\mathcal{R}(T) = \mathcal{R}(A)$: range of $T$ is the range (column) space of $A$

**zero transformation:** $T : \mathbf{R}^n \to \mathbf{R}^m, \quad T(x) = 0$

$$\mathbf{ker}(T) = \mathbb{R}^n, \quad \mathcal{R}(T) = \{0\}$$

**identity operator:** $T : \mathcal{V} \to \mathcal{V}, \quad T(x) = x$

$$\mathbf{ker}(T) = \{0\}, \quad \mathcal{R}(T) = \mathcal{V}$$

**differentiation:** $T : C^1(-\infty, \infty) \to F(-\infty, \infty), \quad T(f) = f'$

$\mathbf{ker}(T)$ is the set of constant functions on $(-\infty, \infty)$



## Rank and Nullity

**Rank** of a linear transformation $T : X \to Y$ is defined as

$$\mathbf{rank}(T) = \dim \mathcal{R}(T)$$

**Nullity** of a linear transformation $T : X \to Y$ is defined as

$$\mathbf{nullity}(T) = \dim \mathbf{ker}(T)$$

(provided that $\mathcal{R}(T)$ and $\mathbf{ker}(T)$ are finite-dimensional)

**Rank-Nullity theorem:** suppose $X$ is a finite-dimensional vector space

$$\mathbf{rank}(T) + \mathbf{nullity}(T) = \dim(X)$$



## Proof of rank-nullity theorem

- assume $\dim(X) = n$
- assume a nontrivial case: $\dim \mathbf{ker}(T) = r$ where $1 < r < n$
- let $\{v_1, v_2, \ldots, v_r\}$ be a basis for $\mathbf{ker}(T)$
- let $W = \{v_1, v_2, \ldots, v_r\} \cup \{v_{r+1}, v_{r+2}, \ldots, v_n\}$ be a basis for $X$
- we can show that
$$S = \{T(v_{r+1}), \ldots, T(v_n)\}$$
forms a basis for $\mathcal{R}(T)$ ($\therefore$ complete the proof since $\dim\ S = n - r$)

$\mathbf{span}\ S = \mathcal{R}(T)$

- for any $z \in \mathcal{R}(T)$, there exists $v \in X$ such that $z = T(v)$
- since $W$ is a basis for $X$, we can represent $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$
- we have $z = \alpha_{r+1} T(v_{r+1}) + \cdots + \alpha_n T(v_n)$ ($\because\ v_1, \ldots, v_r \in \mathbf{ker}(T)$)

$S$ **is linearly independent**, $i.e.$, we must show that

$$\alpha_{r+1}T(v_{r+1}) + \cdots + \alpha_n T(v_n) = 0 \quad \Longrightarrow \quad \alpha_{r+1} = \cdots = \alpha_n = 0$$

- since $T$ is linear

$$\alpha_{r+1}T(v_{r+1}) + \cdots + \alpha_n T(v_n) = T(\alpha_{r+1}v_{r+1} + \cdots + \alpha_n v_n) = 0$$

- this implies $\alpha_{r+1}v_{r+1} + \cdots + \alpha_n v_n \in \mathbf{ker}(T)$

$$\alpha_{r+1}v_{r+1} + \cdots + \alpha_n v_n = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_r v_r$$

- since $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ is linear independent, we must have

$$\alpha_1 = \cdots = \alpha_r = \alpha_{r+1} = \cdots = \alpha_n = 0$$



## One-to-one transformation

a linear transformation $T : X \to Y$ is said to be **one-to-one** if

$$\forall x, z \in X \qquad T(x) = T(z) \quad \Longrightarrow \quad x = z$$

- $T$ never maps distinct vectors in $X$ to the same vector in $Y$
- also known as **injective** tranformation

☞ **Theorem:** $T$ is *one-to-one* if and only if $\mathbf{ker}(T) = \{0\}$, $i.e.$,

$$T(x) = 0 \quad \Longrightarrow \quad x = 0$$

- for $T(x) = Ax$ where $A \in \mathbf{R}^{n \times n}$,

$$T \text{ is one-to-one} \quad \Longleftrightarrow \quad A \text{ is } \textit{invertible}$$



## Onto transformation

a linear transformation $T : X \to Y$ is said to be **onto** if

for **every** vector $y \in Y$, there exists a vector $x \in X$ such that

$$y = T(x)$$

- every vector in $Y$ is the image of at least one vector in $X$
- also known as **surjective** transformation

☞ **Theorem:** $T$ is onto if and only if $\mathcal{R}(T) = Y$

☞ **Theorem:** for a *linear operator* $T : X \to X$,

$T$ is one-to-one if and only if $T$ is onto

✎ which of the following is a one-to-one transformation ?

- $T : \mathbf{P}_n \to \mathbf{R}^{n+1}$

$$T(p(t)) = T(a_0 + a_1 t + \cdots + a_n t^n) = (a_0, a_1, \ldots, a_n)$$

- $T : \mathbf{P}_n \to \mathbf{P}_{n+1}$

$$T(p(t)) = tp(t)$$

- $T : \mathbf{R}^{m \times n} \to \mathbf{R}^{n \times m}, \quad T(X) = X^T$
- $T : \mathbf{R}^{n \times n} \to \mathbf{R}, \quad T(X) = \mathbf{tr}(X)$
- $T : C^1(-\infty, \infty) \to F(-\infty, \infty), \quad T(f) = f'$



## Matrix transformation

consider a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$,

$$T(x) = Ax, \qquad A \in \mathbf{R}^{m \times n}$$

☞ **Theorem:** the following statements are equivalent

- $T$ is **one-to-one**
- the homonegenous equation $Ax = 0$ has only the trivial solution ($x = 0$)
- $\mathbf{rank}(A) = n$

☞ **Theorem:** the following statements are equivalent

- $T$ is **onto**
- for every $b \in \mathbf{R}^m$, the linear system $Ax = b$ always has a solution
- $\mathbf{rank}(A) = m$



## Isomorphism

a linear transformation $T : X \to Y$ is said to be an **isomorphism** if

$T$ is both one-to-one and onto

if there exists an isomorphism between $X$ and $Y$, the two vector spaces are said to be **isomorphic**

☞ **Theorem:**

- for any $n$-dimensional vector space $X$, there always exists a linear transformation $T : X \to \mathbf{R}^n$ that is one-to-one and onto (for example, a coordinate map)
- every real $n$-dimensional vector space is isomorphic to $\mathbf{R}^n$

**examples of isomorphism**

- $T : \mathbf{P}_n \to \mathbf{R}^{n+1}$

$$T(p(t)) = T(a_0 + a_1 t + \cdots + a_n t^n) = (a_0, a_1, \ldots, a_n)$$

  $\mathbf{P}_n$ is isomorphic to $\mathbf{R}^{n+1}$

- $T : \mathbf{R}^{2\times 2} \to \mathbf{R}^4$

$$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4)$$

  $\mathbf{R}^{2\times 2}$ is isomorphic to $\mathbf{R}^4$

in these examples, we observe that

- $T$ maps a vector into its coordinate vector relative to a standard basis
- for any two finite-dimensional vector spaces that are isomorphic, they have the same dimension



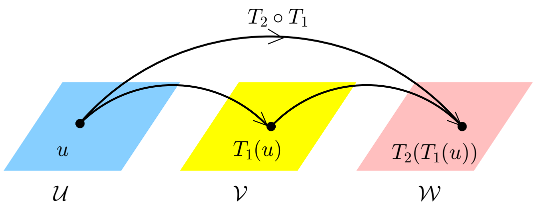
## Composition of linear transformations

let $T_1 : \mathcal{U} \to \mathcal{V}$ and $T_2 : \mathcal{V} \to \mathcal{W}$ be linear transformations

the **composition** of $T_2$ *with* $T_1$ is the function defined by

$$(T_2 \circ T_1)(u) = T_2(T_1(u))$$

where $u$ is a vector in $\mathcal{U}$




**Theorem** ✎ if $T_1, T_2$ are linear, so is $T_2 \circ T_1$



**example 1:** $T_1 : \mathbf{P}_1 \to \mathbf{P}_2$, $\quad T_2 : \mathbf{P}_2 \to \mathbf{P}_2$

$$T_1(p(t)) = tp(t), \quad T_2(p(t)) = p(2t+4)$$

then the composition of $T_2$ with $T_1$ is given by

$$(T_2 \circ T_1)(p(t)) = T_2(T_1(p(t))) = T_2(tp(t)) = (2t+4)p(2t+4)$$

**example 2:** $T : \mathcal{V} \to \mathcal{V}$ is a linear operator, $I : \mathcal{V} \to \mathcal{V}$ is identity operator

$$(T \circ I)(v) = T(I(v)) = T(v), \quad (I \circ T)(v) = I(T(v)) = T(v)$$

hence, $T \circ I = T$ and $I \circ T = T$

**example 3:** $T_1 : \mathbf{R}^n \to \mathbf{R}^m$, $T_2 : \mathbf{R}^m \to \mathbf{R}^n$ with

$$T_1(x) = Ax, \quad T_2(w) = Bw, \quad A \in \mathbf{R}^{m\times n}, B \in \mathbf{R}^{n\times m}$$

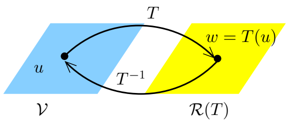
then $T_1 \circ T_2 = AB$ and $T_2 \circ T_1 = BA$

## Inverse of linear transformation

a linear transformation $T : \mathcal{V} \to \mathcal{W}$ is **invertible** if there is a transformation $S : \mathcal{W} \to \mathcal{V}$ satisfying

$$S \circ T = I_{\mathcal{V}} \quad \text{and} \quad T \circ S = I_{\mathcal{W}}$$

we call $S$ the **inverse** of $T$ and denote $S = T^{-1}$




$$T^{-1}(T(u)) = u \quad \forall u \in \mathcal{U}$$
$$T(T^{-1}(w)) = w \quad \forall w \in \mathcal{R}(T)$$

**Facts:**

- if $T$ is one-to-one then $T$ has an inverse
- $T^{-1} : \mathcal{R}(T) \to \mathcal{V}$ is also linear



**example:** $T : \mathbf{R}^n \to \mathbf{R}^n$

$$T(x_1, x_2, \ldots, x_n) = (a_1x_1, a_2x_2, \ldots, a_nx_n)$$

where $a_k \neq 0$ for $k = 1, 2, \ldots, n$

first we show that $T$ is one-to-one, $i.e.$, $T(x) = 0 \Longrightarrow x = 0$

$$T(x_1, \ldots, x_n) = (a_1x_1, \ldots, a_nx_n) = (0, \ldots, 0)$$

this implies $a_kx_k = 0$ for $k = 1, \ldots, n$

since $a_k \neq 0$ for all $k$, we have $x = 0$, or that $T$ is one-to-one

hence, $T$ is invertible and the inverse that can be found from

$$T^{-1}(T(x)) = x$$

which is given by

$$T^{-1}(w_1, w_2, \ldots, w_n) = ((1/a_1)w_1, (1/a_2)w_2, \ldots, (1/a_n)w_n)$$



## Composition of one-to-one linear transformation

if $T_1 : \mathcal{U} \to \mathcal{V}$ and $T_2 : \mathcal{V} \to \mathcal{W}$ are one-to-one linear transformation, then

- $T_2 \circ T_1$ is one-to-one
- $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

**example:** $T_1 : \mathbf{R}^n \to \mathbf{R}^n$, $\quad T_2 : \mathbf{R}^n \to \mathbf{R}^n$

$$\begin{aligned} T_1(x_1, x_2, \ldots, x_n) &= (a_1x_1, a_2x_2, \ldots, a_nx_n), \quad a_k \neq 0, k = 1, \ldots, n \\ T_2(x_1, x_2, \ldots, x_n) &= (x_2, x_3, \ldots, x_n, x_1) \end{aligned}$$

both $T_1$ and $T_2$ are invertible and the inverses are

$$\begin{aligned} T_1^{-1}(w_1, w_2, \ldots, w_n) &= ((1/a_1)w_1, (1/a_2)w_2, \ldots, (1/a_n)w_n) \\ T_2^{-1}(w_1, w_2, \ldots, w_n) &= (w_n, w_1, \ldots, w_{n-1}) \end{aligned}$$

from a direct calculation, the composition of $T_1^{-1}$ with $T_2^{-1}$ is

$$\begin{aligned}(T_1^{-1} \circ T_2^{-1})(w) &= T_1^{-1}(w_n, w_1, \ldots, w_{n-1}) \\ &= ((1/a_1)w_n, (1/a_2)w_1, \ldots, (1/a_n w_{n-1}))\end{aligned}$$

now consider the composition of $T_2$ with $T_1$

$$(T_2 \circ T_1)(x) = (a_2 x_2, \ldots, a_n x_n, a_1 x_1)$$

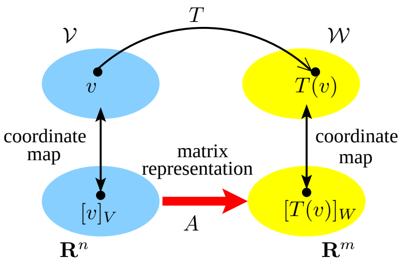
it is clear to see that

$$(T_2 \circ T_1) \circ (T_1^{-1} \circ T_2^{-1}) = I$$



## Matrix representation for linear transformation

let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation




$V$ is a basis for $\mathcal{V}$

$\dim \mathcal{V} = n$

$W$ is a basis for $\mathcal{W}$

$\dim \mathcal{W} = m$

how to represent an image of $T$ in terms of its coordinate vector ?

**Problem:** find a matrix $A \in \mathbf{R}^{m \times n}$ that maps $[v]_V$ into $[T(v)]_W$



**key idea:** the matrix $A$ must satisfy

$$A[v]_V = [T(v)]_W, \quad \textbf{for all} \;\; v \in \mathcal{V}$$

hence, it suffices to hold *for all vector in a basis* for $\mathcal{V}$

suppose a basis for $\mathcal{V}$ is $V = \{v_1, v_2, \ldots, v_n\}$

$$A[v_1] = [T(v_1)], \quad A[v_2] = [T(v_2)], \quad \ldots, \quad A[v_n] = [T(v_n)]$$

(we have dropped the subscripts that refer to the choice of bases $V, W$

$A$ is a matrix of size $m \times n$, so we can write $A$ as

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$$

where $a_k$'s are the columns of $A$

the coordinate vectors of $v_k$'s are simply the standard unit vectors

$$[v_1] = e_1, \quad [v_2] = e_2, \quad \ldots, \quad [v_n] = e_n$$

hence, we have

$$A[v_1] = a_1 = [T(v_1)], \quad A[v_2] = a_2 = [T(v_2)], \quad \cdots, \quad A[v_n] = a_n = [T(v_n)]$$

stack these vectors back in $A$

$$A = \left[\, [T(v_1)] \quad [T(v_2)] \quad \cdots \quad [T(v_n)] \,\right]$$

- the columns of $A$ are the coordinate maps of the images of the basis vectors in $\mathcal{V}$
- we call $A$ the **matrix representation** for $T$ relative to the bases $V$ and $W$ and denote it by
$$[T]_{W,V}$$
- a matrix representation *depends* on the **choice of bases** for $\mathcal{V}$ and $\mathcal{W}$

**special case:** $T : \mathbf{R}^n \to \mathbf{R}^m$, $T(x) = Bx$ we have $[T] = B$ relative to the *standard bases* for $\mathbf{R}^m$ and $\mathbf{R}^n$



**example:** $T : \mathcal{V} \to \mathcal{W}$ where

$$\begin{aligned} \mathcal{V} &= \mathbf{P}_1 \quad \text{with a basis} \quad V = \{1, t\} \\ \mathcal{W} &= \mathbf{P}_1 \quad \text{with a basis} \quad W = \{t-1, t\} \end{aligned}$$

define $T(p(t)) = p(t+1)$, find $[T]$ relative to $V$ and $W$

**solution.**

find the mappings of vectors in $V$ and their coordinates relative to $W$

$$\begin{aligned} T(v_1) = T(1) &= 1 &= -1 \cdot (t-1) + 1 \cdot t \\ T(v_2) = T(t) &= t+1 &= -1 \cdot (t-1) + 2 \cdot t \end{aligned}$$

hence $[T(v_1)]_W = (-1, 1)$ and $[T(v_2)]_W = (-1, 2)$

$$[T]_{WV} = \left[[T(v_1)]_W \quad [T(v_2)]_W\right] = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$



**example:** given a matrix representation for $T : \mathbf{P}_2 \to \mathbf{R}^2$

$$[T] = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix}$$

relative to the bases $V = \{2-t, t+1, t^2-1\}$ and $W = \{(1,0), (1,1)\}$

find the image of $6t^2$ under $T$

**solution.** find the coordinate of $6t^2$ relative to $V$ by writing

$$6t^2 = \alpha_1 \cdot (2-t) + \alpha_2 \cdot (t+1) + \alpha_3 \cdot (t^2-1)$$

solving for $\alpha_1, \alpha_2, \alpha_3$ gives

$$[6t^2]_V = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

from the definition of $[T]$:

$$[T(6t^2)]_W = [T]_{WV}[6t^2]_V = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 30 \end{bmatrix}$$

then we read from $[T(6t^2)]_W$ that

$$T(6t^2) = 8 \cdot (1,0) + 30 \cdot (1,1) = (38,30)$$



## Matrix representation for linear operators

we say $T$ is a **linear operator** if $T$ is a linear transformation from $\mathcal{V}$ to $\mathcal{V}$

- typically we use the same basis for $\mathcal{V}$, says $V = \{v_1, v_2, \ldots, v_n\}$
- a matrix representation for $T$ relative to $V$ is denoted by $[T]_V$ where

$$[T]_V = \begin{bmatrix} [T(v_1)] & [T(v_2)] & \ldots & [T(v_n)] \end{bmatrix}$$

**Theorem**

- $T$ is one-to-one if and only if $[T]_V$ is invertible
- $[T^{-1}]_V = ([T]_V)^{-1}$

what is the matrix (relative to a basis) for the identity operator ?



## Matrix representation for composite transformation

if $T_1 : \mathcal{U} \to \mathcal{V}$ and $T_2 : \mathcal{V} \to \mathcal{W}$ are linear transformations

and $U, V, W$ are bases for $\mathcal{U}, \mathcal{V}, \mathcal{W}$ respectively

then

$$[T_2 \circ T_1]_{W,U} = [T_2]_{W,V} \cdot [T_1]_{V,U}$$

**example:** $T_1 : \mathcal{U} \to \mathcal{V}$, $T_2 : \mathcal{V} \to \mathcal{W}$

$$\mathcal{U} = \mathbf{P}_1, \quad \mathcal{V} = \mathbf{P}_2, \quad \mathcal{W} = \mathbf{P}_3$$

$$U = \{1, t\}, \quad V = \{1, t, t^2\}, \quad W = \{1, t, t^2, t^3\}$$

$$T_1(p(t)) = T_1(a_0 + a_1 t) = 2a_0 - 3a_1 t$$

$$T_2(p(t)) = 3tp(t)$$

find $[T_2 \circ T_1]$

**solution.** first find $[T_1]$ and $[T_2]$

$$\begin{array}{rcl} T_1(1) &=& 2 &=& 2\cdot 1 + 0\cdot t + 0\cdot t^2 \\ T_1(t) &=& -3t &=& 0\cdot 1 - 3\cdot t + 0\cdot t^2 \end{array} \implies [T_1] = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcl} T_2(1) &=& 3t &=& 0\cdot 1 + 3\cdot 1 + 0\cdot t^2 + 0\cdot t^3 \\ T_2(t) &=& 3t^2 &=& 0\cdot 1 + 0\cdot 1 + 3\cdot t^2 + 0\cdot t^3 \\ T_2(t^2) &=& 3t^3 &=& 0\cdot 1 + 0\cdot 1 + 0\cdot t^2 + 3\cdot t^3 \end{array} \implies [T_2] = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

next find $[T_2 \circ T_1]$

$$\begin{array}{rcl} (T_2\circ T_1)(1) &=& T_2(2) &=& 6t \\ (T_2\circ T_1)(t) &=& T_2(-3t) &=& -9t^2 \end{array} \implies [T_2\circ T_1] = \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}$$

easy to verify that $[T_2 \circ T_1] = [T_2]\cdot[T_1]$



## References

Chapter 8 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

## Exercises

1. For each of the following transformations, determine if it is a linear transformation.

   (a) $T$ : $\mathsf{P}_2 \to \mathsf{P}_2$

   $$T(a_0 + a_1 x + a_2 x^2) = (a_0 - a_2) + (a_0 + 2a_1 + a_2)x + (3a_0 - a_2)x^2.$$

   If $T$ is linear, find $\ker(T)$ and its dimension.

   (b) $T$ : $\mathsf{R}^3 \to \mathsf{R}^2$

   $$T(x_1, x_2, x_3) = (6x_1 + x_2 - 3x_3, 4x_1 + x_2 - x_3).$$

   If $T$ is linear, find $\mathcal{R}(T)$ and its dimension.

2. Let $T$ : $\mathsf{R}^n \to \mathsf{R}^n$ be a transformation given by

   $$T(x_1, x_2, \ldots, x_n) = (nx_n, (n-1)x_{n-1}, \ldots, 3x_3, 0, 0)$$

   $T$ sorts the entries of $x$ in the opposite order, and scale the $k$th term by $k$. The last two entries of $T(x)$ are assigned to be zero.

   (a) Show that $T$ is a linear transformation. Hence, $T$ can be represented by $T(x) = Ax$. Determine what $A$ is.

   (b) Is $T$ one-to-one ? If not, find a basis for $\mathbf{ker}(T)$ and its dimension.

   (c) Find a basis for $\mathcal{R}(T)$ and verify the dimension (rank-nullity) theorem.

3. Let $T$ : $\mathsf{R}^{n \times n} \to \mathsf{R}$ be a linear transformation defined by

   $$T(A) = \mathbf{1}^T A \mathbf{1}.$$

   (a) Find $\dim \mathbf{ker}(T)$.

   (b) For $n = 3$, find a basis for $\mathbf{ker}(T)$.

4. Let $T$ : $\mathsf{P}_2 \to \mathsf{P}_2$ be a transformation given by

   $$T(a_0 + a_1 x + a_2 x^2) = (a_0 - a_2) + (a_0 + 2a_1 + a_2)x + (3a_0 - a_2)x^2.$$

   (a) Show that $T$ is a linear operator.

   (b) Is $T$ an isomorphism ?

   (c) Let $V = \{1, x, x^2\}$ and $W = \{-x^2 + 1, 2x + 1, 3x^2\}$ be the two bases for $\mathsf{P}_2$. Find the matrix representation of $T$ relative to $V$ and $W$, $[T]_{W,V}$. In other words, let $p$ be any vector in $\mathsf{P}_2$. The matrix $[T]_{W,V}$ maps $[p]_V$ to $[T(p)]_W$.

   (d) Find the matrix representation of $T$ relative to $V$ and $V$, $[T]_{V,V}$. In other words, we use the same basis for $\mathsf{P}_2$. Is the result the same as in part (c) ? Why ?

   (e) Does the inverse of $T$ exist ? If yes, find $T^{-1}$.

# บทที่ 6

# Eigenvalues and eigenvectors

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถหาค่าเจาะจงและเวคเตอร์เจาะจงของเมทริกซ์จตุรัสได้
- สามารถวิเคราะห์ความสัมพันธ์ของค่าเจาะจงของฟังก์ชันของเมทริกซ์หนึ่งๆ กับค่าเจาะจงของเมทริกซ์นั้นๆ ได้
- สามารถใช้วิธีการแนวทแยง (diagonalization) ในการแปลงเมทริกซ์จตุรัสให้เป็นเมทริกซ์ทแยงได้

EE202 - EE MATH II Jitkomut Songsiri

# 6. Eigenvalues and Eigenvectors

- definition
- important properties
- similarity transform
- diagonalization



## Definition

$\lambda \in \mathbf{C}$ is called an **eigenvalue** of $A \in \mathbf{C}^{n \times n}$ if

$$\det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $x \in \mathbf{C}^n$ s.t. $(\lambda I - A)x = 0$, *i.e.*,

$$Ax = \lambda x$$

any such $x$ is called an **eigenvector** of $A$ (associated with eigenvalue $\lambda$)



## Computing eigenvalues

- $\mathcal{X}(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of $A$
- $\mathcal{X}(\lambda) = 0$ is called the **characteristic equation** of $A$

the characteristic equation provides a way to compute the eigenvalues of $A$

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$$

$$\mathcal{X}(\lambda) = \begin{vmatrix} \lambda - 5 & -3 \\ 6 & \lambda + 4 \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

solving the characteristic equation gives

$$\lambda = 2, -1$$

## Computing eigenvectors

for each eigenvalue of $A$, we can find an associated eigenvector from

$$(\lambda I - A)x = 0$$

where $x$ is a **nonzero** vector

for $A$ in page 6-3, let's find an eigenvector corresponding to $\lambda = 2$

$$(\lambda I - A)x = \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad x_1 + x_2 = 0$$

the equation has many solutions, so we can form the set of solutions by

$$\left\{ x \in \mathbf{R}^2 \;\middle|\; x = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

this set is called the **eigenspace** of $A$ corresponding to $\lambda = 2$



## Eigenspace

eigenspace of $A$ corresponding to $\lambda$ is defined as the nullspace of $\lambda I - A$

$$\mathcal{N}(\lambda I - A)$$

equivalent definition: solution space of the homogeneous system

$$(\lambda I - A)x = 0$$

- an eigenspace is a vector space (by definition)
- $0$ is in every eigenspace but it is not an eigenvector
- the *nonzero* vectors in an eigenspace are the eigenvectors of $A$



from page 6-4, any nonzero vector lies in the eigenspace is an eigenvector of $A$, *e.g.*, $x = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$

same way to find an eigenvector associated with $\lambda = -1$

$$(\lambda I - A)x = \begin{bmatrix} -6 & -3 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad 2x_1 + x_2 = 0$$

so the eigenspace corresponding to $\lambda = -1$ is

$$\left\{ x \;\middle|\; x = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

and $x = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$ is an eigenvector of $A$ associated with $\lambda = -1$

## Properties

- if $A$ is $n \times n$ then $\mathcal{X}(\lambda)$ is a polynomial of order $n$
- if $A$ is $n \times n$ then there are $n$ eigenvalues of $A$
- even when $A$ is real, eigenvalues and eigenvectors can be complex, *e.g.*,

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$$

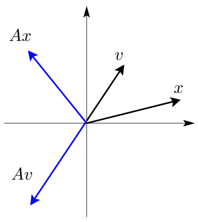
- if $A$ and $\lambda$ are real, we can choose the associated eigenvector to be real
- if $A$ is real then eigenvalues must occur in complex conjugate pairs
- if $x$ is an eigenvector of $A$, so is $\alpha x$ for any $\alpha \in \mathbf{C}$, $\alpha \neq 0$
- an eigenvector of $A$ associated with $\lambda$ lies in $\mathcal{N}(\lambda I - A)$



## Scaling interpretation

assume $\lambda$ is real

if $v$ is an eigenvector, effect of $A$ on $v$ is simple: just scaling by $\lambda$




| | |
|---|---|
| $\lambda > 0$ | $v$ and $Av$ point in same direction |
| $\lambda < 0$ | $v$ and $Av$ point in opposite directions |
| $\lvert\lambda\rvert < 1$ | $Av$ smaller than $v$ |
| $\lvert\lambda\rvert > 1$ | $Av$ larger than $v$ |



## Important facts

denote $\lambda(A)$ an eigenvalue of $A$

- $\lambda(\alpha A) = \alpha\lambda(A)$ for any $\alpha \in \mathbf{C}$
- $\mathbf{tr}(A)$ is the sum of eigenvalues of $A$
- $\det(A)$ is the product of eigenvalues of $A$
- $A$ and $A^T$ share the same eigenvalues ✎
- $\lambda(\overline{A^T}) = \overline{\lambda(A)}$ ✎
- $\lambda(A^m) = (\lambda(A))^m$ for any integer $m$
- $A$ is invertible if and only if $\lambda = 0$ is not an eigenvalue of $A$ ✎

## Matrix powers

the $m$th power of a matrix $A$ for a nonnegative integer $m$ is defined as

$$A^m = \prod_{k=1}^{m} A$$

(the multiplication of $m$ copies of $A$)

and $A^0$ is defined as the identity matrix, $i.e.$, $A^0 = I$

✌ **Facts:** if $\lambda$ is an eigenvalue of $A$ with an eigenvector $v$ then

- $\lambda^m$ is an eigenvalue of $A^m$
- $v$ is an eigenvector of $A^m$ associated with $\lambda^m$



## Invertibility and eigenvalues

$A$ is not invertible if and only if there exists a nonzero $x$ such that

$$Ax = 0, \quad \text{or} \quad Ax = 0 \cdot x$$

which implies $0$ is an eigenvalue of $A$

another way to see this is that

$$A \text{ is not invertible} \iff \det(A) = 0 \iff \det(0 \cdot I - A) = 0$$

which means $0$ is a root of the characteristic equation of $A$

**conclusion** ✎ the following statements are equivalent

- $A$ is invertible
- $\mathcal{N}(A) = \{0\}$
- $\lambda = 0$ is not an eigenvalue of $A$



## Eigenvalues of special matrices

**diagonal matrix:**

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

eigenvalues of $D$ are the diagonal elements, $i.e.$, $\lambda = d_1, d_2, \ldots, d_n$

**triangular matrix:**

upper triangular lower triangular

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

eigenvalues of $L$ and $U$ are the diagonal elements, $i.e.$, $\lambda = a_{11}, \ldots, a_{nn}$

## Similarity transform

two $n \times n$ matrices $A$ and $B$ are said to be **similar** if

$$B = T^{-1}AT$$

for some invertible matrix $T$

$T$ is called a **similarity transform**

☞ **invariant** properties under similarity transform:

- $\det(B) = \det(A)$
- $\mathbf{tr}(B) = \mathbf{tr}(A)$
- $A$ and $B$ have the same eigenvalues

$$\det(\lambda I - B) = \det(\lambda T^{-1}T - T^{-1}AT) = \det(\lambda I - A)$$



## Diagonalization

an $n \times n$ matrix $A$ is **diagonalizable** if there exists $T$ such that

$$T^{-1}AT = D$$

is *diagonal*

- similarity transform by $T$ *diagonalizes* $A$
- $A$ and $D$ are similar, so the entries of $D$ must be the eigenvalues of $A$

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- computing $A^k$ is simple because $A^k = (TDT^{-1})^k = TD^kT^{-1}$



how to find a matrix $T$ that diagonalizes $A$ ?

suppose $\{v_1, \ldots, v_n\}$ is a *linearly independent* set of eigenvectors of $A$

$$Av_i = \lambda_i v_i \quad i = 1, \ldots, n$$

we can express this equation in the matrix form as

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, so

$$AT = TD$$

since $T$ is invertible ($v_1, \ldots, v_n$ are independent), finally we have

$$T^{-1}AT = D$$

conversely, if there exists $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ that diagonalizes $A$

$$T^{-1}AT = D = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

then $AT = TD$, or

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

so $\{v_1, \dots, v_n\}$ is a linearly independent set of eigenvectors of $A$

**conclusion:** $A$ is diagonalizable *if and only if*

$n$ eigenvectors of $A$ are linearly independent
(eigenvectors form a basis for $\mathbf{C}^n$)

- a diagonalizable matrix is called a **simple** matrix
- if $A$ is not diagonalizable, sometimes it is called *defective*



## Example

find $T$ that diagonalizes

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

the characteristic equation is

$$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

the eigenvalues of $A$ are $\lambda = 5, 3, 3$

an eigenvector associated with $\lambda_1 = 5$ can be found by

$$(5 \cdot I - A)x = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 \text{ is a free variable} \end{array}$$

an eigenvector is $v_1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$



next, find an eigenvector associated with $\lambda_2 = 3$

$$(3 \cdot I - A)x = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_1 + x_3 = 0 \\ x_2, x_3 \text{ are free variables} \end{array}$$

the eigenspace can be written by

$$\left\{ x \;\middle|\; x = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

hence we can find two *independent* eigenvectors

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to the repeated eigenvalue $\lambda_2 = 3$

easy to show that $v_1, v_2, v_3$ are linearly independent

we form a matrix $T$ whose columns are $v_1, v_2, v_n$

$$T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

then $v_1, v_2, v_3$ are linearly independent if and only if $T$ is invertible

by a simple calculation, $\det(T) = 2 \neq 0$, so $T$ is invertible

hence, we can use this $T$ to diagonalize $A$ and it is easy to verify that

$$T^{-1}AT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



## Not all matrices are diagonalizable

**example:** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

characteristic polynomial is $\det(\lambda I - A) = s^2$, so $0$ is the only eigenvalue

eigenvector satisfies $Ax = 0 \cdot x$, $i.e.$,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_2 = 0 \\ x_1 \text{ is a free variable} \end{array}$$

so all eigenvectors has form $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ where $x_1 \neq 0$

thus $A$ cannot have *two independent* eigenvectors



## Distinct eigenvalues

**Theorem:** if $A$ has distinct eigenvalues, $i.e.$,

$$\lambda_i \neq \lambda_j, \quad i \neq j$$

then a set of corresponding eigenvectors are *linearly independent*

which further implies that $A$ is diagonalizable

the converse is *false* – $A$ can have repeated eigenvalues but still be diagonalizable

example: all eigenvalues of $I$ are $1$ (repeated eigenvalues) but $I$ is diagonal

**Proof by contradiction:** assume the eigenvectors are dependent

(simple case) let $Ax_k = \lambda_k x_k$, $k = 1, 2$

suppose there exists $\alpha_1, \alpha_2 \neq 0$

$$\alpha_1 x_1 + \alpha_2 x_2 = 0 \qquad (1)$$

multiplying (1) by $A$: $\quad \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 = 0$

multiplying (1) by $\lambda_1$: $\quad \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_1 x_2 = 0$

subtracting the above from the previous equation

$$\alpha_2(\lambda_2 - \lambda_1)x_2 = 0$$

since $\lambda_1 \neq \lambda_2$, we must have $\alpha_2 = 0$ and consequently $\alpha_1 = 0$

the proof for a general case is left as an exercise



## Eigenvalues of symmetric matrices

$A$ is an $n \times n$ (real) symmetric matrix, $i.e.$, $A = A^T$

$x^*$ denotes $\bar{x}^T$ (complex conjugate transpose)

**Facts** ✌

- $y^*Ay$ is real for all $y \in \mathbf{C}^n$
- all eigenvalues of $A$ are real
- eigenvectors with distinct eigenvalues are orthogonal, i.e.,

$$\lambda_j \neq \lambda_k \quad \Longrightarrow \quad x_j^T x_k = 0$$

- there exists an *orthogonal* matrix $U$ ($U^TU = UU^T = I$) such that

$$A = UDU^T$$

  (symmetric matrices are *always* diagonalizable)



## MATLAB commands

`[V,D] = eig(A)` produces a diagonal matrix `D` of eigenvalues and a full matrix `V` whose columns are the corresponding eigenvectors

```
>> A = [5 3;-6 -4];
>> [V,D] = eig(A)
V =
    0.7071   -0.4472
   -0.7071    0.8944
D =
     2     0
     0    -1
```

$\lambda_1 = 2$ and $\lambda_2 = -1$ and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 0.7071 & 0.7071 \end{bmatrix}^T, \quad v_2 = \begin{bmatrix} -0.4472 & 0.8944 \end{bmatrix}^T$$

note that the eigenvector is normalized so that it has unit norm

**power of a matrix**: use ^ to compute a power of $A$

```
>> A^3
ans =
    17     9
   -18   -10

>> eig(A^3)
ans =
     8
    -1

>> V*D^3*inv(V)
ans =
    17     9
   -18   -10
```

agree with the fact that the eigenvalue of $A^3$ is $\lambda^3$ and $A^3 = TD^3T^{-1}$



## References

Chapter 5 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

Lecture note on

*Linear algebra*, EE263, S. Boyd, Stanford university

## Exercises

1. True/False questions. For each of the following statements, either show that it is true, or give a specific counterexample.

   (a) If $A$ is real, then $\lambda$ is real.

   (b) If one of the eigenvalues of $A$ is complex, then $A$ must be complex.

   (c) $A$ and $A^T$ share the same eigenvalues.

   (d) $A$ and $A^T$ share the same eigenvector corresponding to the same eigenvalue.

   (e) $I + A$ is always invertible, even if $A$ is not invertible.

   (f) If $A$ is similar to $B$, then $2I + 3A$ is similar to $2I + 3B$.

   (g) If $(\lambda_1, x_1)$ and $(\lambda_2, x_2)$ are any two eigenvalue/eigenvector pairs of $A$, then $\lambda_1 + \lambda_2$ is an eigenvalue of $A$, associated with an eigenvector $x_1 + x_2$.

   (h) If $A$ is diagonalizable, then there exists a unique matrix $T$ such that $T^{-1}AT$ is diagonal.

   (i) If $A$ and $B$ are similar invertible matrices, then $A^{-1}$ and $B^{-1}$ are similar.

   (j) If $A$ is similar to $\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, then $2$ and $-3$ are eigenvalues of $A$.

   (k) If one of the eigenvalues of $A$ is zero, then $A$ is not row equivalent to the identity matrix.

   (l) If A is invertible, then A is diagonalizable.

2. Let
$$A = \begin{bmatrix} -5 & -3 & 3 \\ 2 & 0 & -2 \\ -4 & -4 & 2 \end{bmatrix}.$$

   (a) Find all the eigenvalues and corresponding eigenvectors of $A$.

   (b) Is $A$ invertible ? Justify your answer. If $A$ is invertible, find the eigenvalues of $5A^{-1}$.

   (c) Find the eigenvalues of $(A + 3I)(A - 2I)$.

   (d) Determine if $A$ diagonalizable and justify your answer. If $A$ is diagonalizable, use the diagonalization technique to compute $A^2 + A - 6I$.

3. Let
$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

   Find $M = \det[(3A - I)^{20}(4A + I)^T(-A + 2I)^{-1}]$.

# บทที่ 7

# Functions of Square Matrices

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- รู้จักนิยามและคุณสมบัติของฟังก์ชันพื้นฐานของเมทริกซ์จตุรัส เช่นฟังก์ชันพหุนาม เป็นต้น
- สามารถประยุกต์ใช้เทคนิค diagonalization หรือใช้ทฤษฎีบทเคย์เลย์ฮามิลตัน (Cayley-Hamilton Theorem) ในการหาฟังก์ชันของเมทริกซ์จตุรัสได้
- สามารถแก้สมการอนุพันธ์สามัญเชิงเส้น (linear ordinary differential equations) ด้วยการประยุกต์ใช้งานฟังก์ชันของเมทริกซ์จตุรัสได้

# 7. Function of square matrices

- matrix polynomial
- rational function
- Cayley-Hamilton theotem
- infinite series
- matrix exponential
- applications to differential equations



## Matrix Power

the $m$th power of a matrix $A$ for a *nonnegative* $m$ is defined as

$$A^m = \prod_{k=1}^{m} A$$

and define $A^0 = I$

**property:** $A^r A^s = A^s A^r = A^{r+s}$

a *negative* power of $A$ is defined as

$$A^{-n} = (A^{-1})^n$$

$n$ is a nonnegative integer and $A$ is invertible



## Matrix polynomial

a **matrix polynomial** is a polynomial with matrices as variables

$$p(A) = a_0 I + a_1 A + \cdots + a_n A^n$$

for example $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

$$\begin{aligned} p(A) = 2I - 6A + 3A^2 &= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 6\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 2 & -3 \\ 0 & 11 \end{bmatrix} \end{aligned}$$

**Fact** ✎ any two polynomials of $A$ commute, $i.e.$, $p(A)q(A) = q(A)p(A)$

**similarity transform:** suppose $A$ is diagonalizable, $i.e.$,

$$\Lambda = T^{-1}AT \quad \Longleftrightarrow \quad A = T\Lambda T^{-1}$$

where $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, $i.e.$, the columns of $T$ are eigenvectors of $A$

then we have $A^k = T\Lambda^k T^{-1}$

thus diagonalization simplifies the expression of a matrix polynomial

$$\begin{aligned} p(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\ &= a_0 TT^{-1} + a_1 T\Lambda T^{-1} + \cdots + a_n T\Lambda^n T^{-1} \\ &= Tp(\Lambda)T^{-1} \end{aligned}$$

where

$$p(\Lambda) = \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix}$$



**eigenvalues and eigenvectors** ✌

if $\lambda$ and $v$ be an eigenvalue and corresponding eigenvector of $A$ then

- $p(\lambda)$ is an eigenvalue of $p(A)$
- $v$ is a corresponding eigenvector of $p(A)$

$$Av = \lambda v \quad \Longrightarrow \quad A^2 v = \lambda A v = \lambda^2 v \quad \cdots \quad \Longrightarrow \quad A^k v = \lambda^k v$$

thus

$$(a_0 I + a_1 A + \cdots + a_n A^n)v = (a_0 v + a_1 \lambda + \cdots + a_n \lambda^n)v$$

which shows that

$$p(A)v = p(\lambda)v$$



## Rational functions

$f(x)$ is called a **rational function** if and only if it can be written as

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions in $x$ and $q(x) \neq 0$

we define a rational function for square matrices as

$$f(A) = \frac{p(A)}{q(A)} \;\triangleq\; p(A)q(A)^{-1} = q^{-1}(A)p(A)$$

provided that $q(A)$ is invertible

**eigenvalues and eigenvectors** ☞

if $\lambda$ and $v$ be an eigenvalue and corresponding eigenvector of $A$ then

- $p(\lambda)/q(\lambda)$ is an eigenvalue of $f(A)$
- $v$ is a corresponding eigenvector of $f(A)$

both $p(A)$ and $q(A)$ are polynomials, so we have

$$p(A)v = p(\lambda)v, \quad q(A)v = q(\lambda)v$$

and the eigenvalue of $q(A)^{-1}$ is $1/q(\lambda)$, *i.e.*, $q(A)^{-1}v = (1/q(\lambda))v$

thus

$$f(A)v = p(A)q(A)^{-1}v = q(\lambda)^{-1}p(A)v = q(\lambda)^{-1}p(\lambda)v = f(\lambda)v$$

which says that $f(\lambda)$ is an eigenvalue of $f(A)$ with the same eigenvector



**example:** $f(x) = (x+1)/(x-5)$ and $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$

$$\det(\lambda I - A) = 0 = (\lambda - 4)(\lambda - 5) - 2 = \lambda^2 - 9\lambda + 18 = 0$$

the eigenvalues of $A$ are $\lambda_1 = 3$ and $\lambda_2 = 6$

$$f(A) = (A+I)(A-5I)^{-1} = \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$$

the characteristic function of $f(A)$ is

$$\det(\lambda I - f(A)) = 0 = (\lambda - 1)(\lambda - 4) - 18 = \lambda^2 - 5\lambda - 14 = 0$$

the eigenvalues of $f(A)$ are $7$ and $-2$

this agrees to the fact that the eigenvalues of $f(A)$ are

$$f(\lambda_1) = (\lambda_1 - 1)/(\lambda_1 - 5) = -2, \quad f(\lambda_2) = (\lambda_2 - 1)/(\lambda_2 - 5) = 7$$



## Cayley-Hamilton theorem

the characteristic polynomial of a matrix $A$ of size $n \times n$

$$\mathcal{X}(\lambda) = \det(\lambda I - A)$$

can be written as a polynomial of degree $n$:

$$\mathcal{X}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$$

☞ **Theorem:** a square matrix satisfies its characteristic equation:

$$\mathcal{X}(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = 0$$

**result:** for $m \geq n$, $A^m$ is a linear combination of $A^k, k = 0, 1, \ldots, n-1$.

**example 1:** $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ the characteristic equation of $A$ is

$$\mathcal{X}(\lambda) = (\lambda - 1)(\lambda - 3) = \lambda^2 - 4\lambda + 3 = 0$$

the Cayley-Hamilton states that $A$ *satisfies its characteristic equation*

$$\mathcal{X}(A) = A^2 - 4A + 3I = 0$$

use this equation to write matrix powers of $A$

$$\begin{array}{lllllll} A^2 & = & 4A - 3I & & & & \\ A^3 & = & 4A^2 - 3A & = & 4(4A - 3I) - 3A & = & 13A - 12I \\ A^4 & = & 13A^2 - 12A & = & 13(4A - 3I) - 12A & = & 40A - 39I \\ \vdots & & & & \vdots & & \end{array}$$

powers of $A$ can be written as a linear combination of $I$ and $A$



**example 2:** with $A$ in page 7-10, find the closed-form expression of $A^k$

for $k \geq 2$, $A^k$ is a linear combination of $I$ and $A$, $i.e.$,

$$A^k = \alpha_0 I + \alpha_1 A$$

where $\alpha_1, \alpha_0$ are to be determined

multiply eigenvectors of $A$ on both sides

$$\begin{array}{lllll} A^k v_1 & = & (\alpha_0 I + \alpha_1 A) v_1 & \Rightarrow & \lambda_1^k = \alpha_0 + \alpha_1 \lambda_1 \\ A^k v_2 & = & (\alpha_0 I + \alpha_1 A) v_2 & \Rightarrow & \lambda_2^k = \alpha_0 + \alpha_1 \lambda_2 \end{array}$$

substitute $\lambda_1 = 1$ and $\lambda_2 = 3$ and solve for $\alpha_0, \alpha_1$

$$\begin{bmatrix} 1 \\ 3^k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \quad \Rightarrow \quad \alpha_0 = \frac{3 - 3^k}{2}, \quad \alpha_1 = \frac{3^k - 1}{2}$$

$$A^k = \frac{3 - 3^k}{2} I + \frac{3^k - 1}{2} A = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}, \quad k \geq 2$$



## Computing the inverse of a matrix

$A$ is a square matrix with the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

by the C-H theorem, $A$ satisfies the characteristic equation

$$A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

if $A$ is invertible, multiply $A^{-1}$ on both sides

$$A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I + a_0 A^{-1} = 0$$

thus the inverse of $A$ can be alternatively computed by

$$A^{-1} = -\frac{1}{a_0} \left( A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I \right)$$

**example:** given $A = \begin{bmatrix} 2 & -4 & -4 \\ 1 & -4 & -5 \\ 1 & 4 & 5 \end{bmatrix}$ find $A^{-1}$

the characteristic equation of $A$ is

$$\det(\lambda I - A) = \lambda^3 - 3\lambda^2 + 10\lambda - 8 = 0$$

$0$ is not an eigenvalue of $A$, so $A$ is invertible and given by

$$\begin{aligned} A^{-1} &= \frac{1}{8}\left(A^2 - 3A + 10I\right) \\ &= \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ -5 & 7 & 3 \\ 4 & -6 & 2 \end{bmatrix} \end{aligned}$$

compare the result with other methods



## Infinite series

**Definition:** a series $\sum_{k=0}^{\infty} a_k$ *converges* to $S$ if the partial sum

$$S_n \triangleq \sum_{k=0}^{n} a_k$$

converges to $S$ as $n \to \infty$

**example of convergent series:**

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots &= 2 \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \log(2) \end{aligned}$$



## Power series

a power series in scalar variable $z$ is an infinite series of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

**example:** power series that converges for *all values* of $z$

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ \cosh(z) &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \\ \sinh(z) &= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \end{aligned}$$

## Power series of matrices

let $A$ be matrix and $A_{ij}$ denotes the $(i,j)$ entry of $A$

**Definition:** a matrix power series

$$\sum_{k=0}^{\infty} a_k A_k$$

*converges* to $S$ if all $(i,j)$ entries of the partial sum

$$S_n \triangleq \sum_{k=0}^{n} a_k A_k$$

converges to the corresponding $(i,j)$ entries of $S$ as $n \to \infty$

**Fact** ☞ if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a convergent power series for *all* $z$ then

$f(A)$ is convergent for *any square matrix* $A$



## Matrix exponential

generalize the exponential function of a scalar

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

to an exponential function of a matrix

define **matrix exponential** as

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \quad = \quad \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

for a square matrix $A$

the infinite series converges for all $A$



**example:** $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

find all powers of $A$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ldots, \quad A^k = A \quad \text{for } k = 2, 3, \ldots$$

so by definition,

$$e^A \;=\; \sum_{k=0}^{\infty} \frac{A^k}{k!} \;=\; I + \sum_{k=1}^{\infty} \frac{A^k}{k!} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \;=\; \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

never compute $e^A$ by element-wise operation !

$$e^A \neq \begin{bmatrix} e^1 & e^1 \\ e^0 & e^0 \end{bmatrix}$$

## Eigenvalues of matrix exponential

☞ if $\lambda$ and $v$ be an eigenvalue and corresponding eigenvector of $A$ then

- $e^\lambda$ is an eigenvalue of $e^A$
- $v$ is a corresponding eigenvector of $e^A$

since $e^A$ can be expressed as power series of $A$:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

multiplying $v$ on both sides and using $A^k v = \lambda^k v$ give

$$\begin{aligned} e^A v &= v + Av + \tfrac{A^2 v}{2!} + \tfrac{A^3 v}{3!} + \cdots \\ &= \left(1 + \lambda + \tfrac{\lambda^2}{2!} + \tfrac{\lambda^3}{3!} + \cdots\right) v \\ &= e^\lambda v \end{aligned}$$



## Properties of matrix exponential

- $e^0 = I$
- $e^{A+B} \neq e^A \cdot e^B$
- if $AB = BA$, *i.e.*, $A$ and $B$ commute, then $e^{A+B} = e^A \cdot e^B$
- $(e^A)^{-1} = e^{-A}$

☞ these properties can be proved by the definition of $e^A$



## Computing $e^A$ via diagonalization

if $A$ is diagonalizable, *i.e.*,

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where $\lambda_k$'s are eigenvalues of $A$ then $e^A$ has the form

$$e^A = Te^\Lambda T^{-1}$$

- computing $e^\Lambda$ is simple since $\Lambda$ is diagonal
- one needs to find eigenvectors of $A$ to form the matrix $T$
- the expression of $e^A$ follows from

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(T\Lambda T^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{T\Lambda^k T^{-1}}{k!} = Te^\Lambda T^{-1}$$

- if $A$ is diagonalizable, so is $e^A$

**example:** compute $f(A) = e^A$ given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

the eigenvalues and eigenvectors of $A$ are

$$\lambda_1 = 1,\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2,\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_3 = 0,\ v_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

form $T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ and compute $e^A = Te^{\Lambda}T^{-1}$

$$e^A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e & e^2 - e & (e^2 - 2e + 1)/2 \\ 0 & e^2 & (e^2 - 1)/2 \\ 0 & 0 & 1 \end{bmatrix}$$



## Computing $e^A$ via C-H theorem

$e^A$ is an infinite series

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

by C-H theorem, the power $A^k$ can be written as

$$A^k = a_0 I + a_1 A + \cdots + a_{n-1} A^{n-1}, \quad k = n, n+1, \ldots$$

(a polynomial in $A$ of order $\leq n - 1$)

thus $e^A$ can be expressed as a linear combination of $I, A, \ldots, A^{n-1}$

$$e^A = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

where $\alpha_k$'s are coefficients to be determined



this also holds for any convergent power series $f(A) = \sum_{k=0}^{\infty} a_k A^k$

$$f(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}$$

(recursively write $A^k$ as a linear combination of $I, A, \ldots, A^{n-1}$ for $k \geq n$)

multiplying an eigenvector $v$ of $A$ on both sides and using $v \neq 0$, we get

$$f(\lambda) = \alpha_0 I + \alpha_1 \lambda + \cdots + \alpha_{n-1} \lambda^{n-1}$$

substitute with the $n$ eigenvalues of $A$

$$\begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \vdots \\ f(\lambda_n) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$

**Fact** ✎ if all $\lambda_k$'s are distinct, the system is solvable and has a unique sol.

## Vandermonde matrix

a *Vandermonde matrix* has the form

$$V = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

(with a geometric progression in each row)

✎ one can show that the determinant of $V$ can be expressed as

$$\det(V) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$$

hence, $V$ is invertible as long as $\lambda_i$'s are *distinct*



**example:** compute $f(A) = e^A$ given

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the eigenvalues of $A$ are $\lambda = 1, 2, 0$ (all are distinct)

form a system of equations: $f(\lambda_i) = \alpha_0 + \alpha_1 \lambda_i + \alpha_2 \lambda_i^2$ for $i = 1, 2, 3$

$$\begin{bmatrix} e^1 \\ e^2 \\ e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

which has the solution

$$\alpha_0 = 1, \quad \alpha_1 = 2e - e^2/2 - 3/2, \quad \alpha_2 = -e + e^2/2 + 1/2$$



substituting $\alpha_0, \alpha_1, \alpha_2$ in

$$e^A = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

gives

$$\begin{aligned} e^A &= \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_0 + \alpha_1 + \alpha_2 & \alpha_1 + 3\alpha_2 & \alpha_2 \\ 0 & \alpha_0 + 2\alpha_1 + 4\alpha_2 & \alpha_1 + 2\alpha_2 \\ 0 & 0 & \alpha_0 \end{bmatrix} \\ &= \begin{bmatrix} e & e^2 - e & (e^2 - 2e + 1)/2 \\ 0 & e^2 & (e^2 - 1)/2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(agree with the result in page 7-22)

## Repeated eigenvalues

$A$ has repeated eigenvalues, *i.e.*, $\lambda_i = \lambda_j$ for some $i, j$

**goal:** compute $f(A)$ using C-H theorem

however, we can no longer apply the result in page 7-24 because

- the number of independent equations on page 7-24 is less than $n$
- the Vandermonde matrix (page 7-25) is not invertible

cannot form a linear system to solve for the $n$ coefficients, $\alpha_0, \ldots, \alpha_{n-1}$



**solution:** for the repeated root with multiplicity $r$

get $r-1$ independent equations by taking derivatives on $f(\lambda)$ w.r.t $\lambda$

$$\begin{aligned}
f(\lambda) &= \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1} \\
\frac{df(\lambda)}{d\lambda} &= \alpha_1 + 2\alpha_2\lambda + \cdots + (n-1)\alpha_{n-1}\lambda^{n-2} \\
\vdots &= \vdots \\
\frac{d^{r-1}f(\lambda)}{d^{r-1}\lambda} &= (r-1)!\alpha_{r-1} + \cdots + (n-r)\cdots(n-2)(n-1)\alpha_{n-1}\lambda^{n-1-r}
\end{aligned}$$



**example:** compute $f(A) = \cos(A)$ given

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the eigenvalues of $A$ are $\lambda_1 = 1, 1$ and $\lambda_2 = 2$

by C-H theorem, write $f(A)$ as a linear combination of $A^k$, $k = 0, ..., n-1$

$$f(A) = \cos(A) = \alpha_0 + \alpha_1 A + \alpha_2 A^2$$

the eigenvalues of $A$ must also satisfies this equation

$$f(\lambda) = \cos(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$$

the derivative of $f$ w.r.t $\lambda$ is given by

$$f'(\lambda) = -\sin(\lambda) = \alpha_1 + 2\alpha_2\lambda$$

thus we can obtain $n$ linearly independent equations:

$$\begin{bmatrix} f(\lambda_1) \\ f'(\lambda_1) \\ f(\lambda_2) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 0 & 1 & 2\lambda_1 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \cos(1) \\ -\sin(1) \\ \cos(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}$$

which have the solution

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2\sin(1) + \cos(2) \\ 2\cos(1) - 3\sin(1) - 2\cos(2) \\ -\cos(1) + \sin(1) + \cos(2) \end{bmatrix}$$

substitute $\alpha_0, \alpha_1, \alpha_2$ to obtain $f(A)$

$$\begin{aligned} f(A) = \cos(A) &= \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \\ &= \begin{bmatrix} \cos(1) & -\sin(1) & 0 \\ 0 & \cos(1) & 0 \\ 0 & 0 & \cos(2) \end{bmatrix} \end{aligned}$$



## Applications to ordinary differential equations

we solve the following first-order ODEs for $t \geq 0$ where $x(0)$ is given

**scalar:** $x(t) \in \mathbf{R}$ and $a \in \mathbf{R}$ is given

$$\dot{x}(t) = ax(t)$$

**solution:** $x(t) = e^{at}x(0)$, for $t \geq 0$

**vector:** $x(t) \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$ is given

$$\dot{x}(t) = Ax(t)$$

**solution:** $x(t) = e^{At}x(0)$, for $t \geq 0$ (use $\dfrac{de^{At}}{dt} = Ae^{At} = e^{At}A$)



## Applications to difference equations

we solve the difference equations for $t = 0, 1, \ldots$ where $x(0)$ is given

**scalar:** $x(t) \in \mathbf{R}$ and $a \in \mathbf{R}$ is given

$$x(t+1) = ax(t)$$

**solution:** $x(t) = a^t x(0)$, for $t = 0, 1, 2, \ldots$

**vector:** $x(t) \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$ is given

$$x(t+1) = Ax(t)$$

**solution:** $x(t) = A^t x(0)$, for $t = 0, 1, 2, \ldots$

**example:** solve the ODE

$$\ddot{y}(t) - \dot{y}(t) - 6y(t) = 0, \quad y(0) = 1, \dot{y}(0) = 0$$

**solution:** define

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

write the equation into the vector form $\dot{x}(t) = Ax(t)$

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \dot{y}(t) + 6y(t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ 6x_1(t) + x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x(t) \end{aligned}$$

the initial condition is

$$x(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



thus it is left to compute $e^{At}$

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

the eigenvalues and eigenvectors of $A$ are

$$\lambda_1 = -2, \ v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_2 = 3, \ v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

all eigenvalues are distinct, so $A$ is diagonalizable and

$$e^{At} = Te^{\Lambda t}T^{-1}, \quad T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$



the closed-form expression of $e^{At}$ is

$$e^{At} = \frac{1}{5} \begin{bmatrix} 3e^{-2t} + 2e^{3t} & -e^{-2t} + e^{3t} \\ -6e^{-2t} + 6e^{3t} & 2e^{-2t} + 3e^{3t} \end{bmatrix}$$

the solution to the vector equation is

$$\begin{aligned} x(t) = e^{At}x(0) &= \frac{1}{5} \begin{bmatrix} 3e^{-2t} + 2e^{3t} & -e^{-2t} + e^{3t} \\ -6e^{-2t} + 6e^{3t} & 2e^{-2t} + 3e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 3e^{-2t} + 2e^{3t} \\ -6e^{-2t} + 6e^{3t} \end{bmatrix} \end{aligned}$$

hence the solution $y(t)$ can be obtained by

$$y(t) = x_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = \frac{1}{5} \left( 3e^{-2t} + 2e^{3t} \right), \quad t \geq 0$$

**example:** solve the difference equation

$$y(t+2) - y(t+1) - 6y(t) = 0, \quad y(0) = 1, y(1) = 0$$

**solution:** define

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix}$$

write the equation into the vector form $x(t+1) = Ax(t)$

$$\begin{aligned} x(t+1) &= \begin{bmatrix} y(t+1) \\ y(t+2) \end{bmatrix} = \begin{bmatrix} y(t+1) \\ y(t+1) + 6y(t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ 6x_1(t) + x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x(t) \end{aligned}$$

the initial condition is

$$x(0) = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



thus it is left to compute $A^t$

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

the eigenvalues and eigenvectors of $A$ are

$$\lambda_1 = -2,\ v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_2 = 3,\ v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

all eigenvalues are distinct, so $A$ is diagonalizable and

$$A^t = T\Lambda^t T^{-1}, \quad T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} (-2)^t & 0 \\ 0 & 3^t \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$



the closed-form expression of $A^t$ is

$$A^t = \frac{1}{5} \begin{bmatrix} 2(3^t) + 3(-2)^t & 3^t - (-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} & 3^{t+1} - (-2)^{t+1} \end{bmatrix}$$

for $t = 0, 1, 2, \ldots$

the solution to the vector equation is

$$\begin{aligned} x(t) = A^t x(0) &= \frac{1}{5} \begin{bmatrix} 2(3^t) + 3(-2)^t & 3^t - (-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} & 3^{t+1} - (-2)^{t+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2(3^t) + 3(-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} \end{bmatrix} \end{aligned}$$

hence the solution $y(t)$ can be obtained by

$$y(t) = x_1(t) = \frac{1}{5} \left( 2(3^t) + 3(-2)^t \right), \quad t = 0, 1, 2, \ldots$$

## MATLAB commands

- `expm(A)` computes the matrix exponential $e^A$
- `exp(A)` computes the exponential of the entries in $A$

example from page 7-18, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$

```
>> A=[1 1;0 0];
>> expm(A)
ans =
    2.7183    1.7183
         0    1.0000
>> exp(A)
ans =
    2.7183    2.7183
    1.0000    1.0000
```



## References

Chapter 21 in

M. Dejnakarin, Mathematics for Electrical Engineers, 3rd edition, Chulalongkorn University Press, 2006

Lecture note on

*Linear algebra*, EE263, S. Boyd, Stanford university

## Exercises

1. Given

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

   Answer the following questions.

   (a) Is $A$ diagonalizable ? Explain in details.

   (b) For any integer $k > 3$, find $A^k w$ where $w = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Express your answer as a vector whose entries are functions of $k$.

2. Use a matrix exponential to solve the system of differential equations:

$$\begin{aligned} \dot{x}_1(t) &= -3x_1(t) + x_2(t) + x_3(t), \\ \dot{x}_2(t) &= x_1(t) - 3x_2(t) + x_3(t), \\ \dot{x}_3(t) &= x_1(t) + x_2(t) - 3x_3(t) \end{aligned}$$

   where $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1$. Show your calculation in details. Check the expression of the matrix exponential of this system with MATLAB by using the command:

```
>> syms t  % to define 't' as a symbolic variable
>> expm(A*t)  % to compute the matrix exponential
```

# บทที่ 8

# Complex Numbers

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถหาค่าอาร์กิวเมนต์ (argument) และค่าอาร์กิวเมนต์มุขสำคัญ (principal value of argument) ของจำนวนเชิงซ้อนได้
- สามารถหารากที่ $n$ ของจำนวนเชิงซ้อนได้
- สามารถอธิบายนิยามที่สำคัญของเซตย่อยในระนาบเชิงซ้อน เช่น เซตเปิด เซตปิด เซตมีขอบเขต เซตต่อกัน (connected set) ได้

EE202 - EE MATH II Jitkomut Songsiri

# 8. Complex Numbers

- sums and products
- basic algebraic properties
- complex conjugates
- exponential form
- principal arguments
- roots of complex numbers
- regions in the complex plane



## Introduction

we denote a complex number $z$ by

$$z = x + jy$$

where

- $x = \mathrm{Re}(z)$ (real part of $z$)
- $y = \mathrm{Im}(z)$ (imaginary part of $z$)
- $j = \sqrt{-1}$



## Sum and Product

consider two complex numbers

$$z_1 = x_1 + jy_1, \quad z_2 = x_2 + jy_2$$

the sum and product of two complex number are defined as:

- $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ addition
- $z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(y_1 x_2 + x_1 y_2)$ multiplication

**example:**

$$(-3 + j5)(1 - 2j) = 7 + j11$$

## Algebraic properties ✎

- $z_1 = z_2 \Longleftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$ equality
- $z_1 + z_2 = z_2 + z_1$ commutative
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ associative
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ distributive
- $-z = -x - jy$ addtive inverse
- $z^{-1} = \dfrac{x}{x^2 + y^2} - j\dfrac{y}{x^2 + y^2}$ multiplicative inverse



## Complex conjugate and Moduli ✎

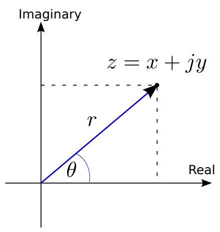
**modulus** (or absolute value): $|z| = \sqrt{x^2 + y^2}$

**complex conjugate:** $\bar{z} = x - jy$

- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ triangle inequality
- $|z_1 + z_2| \geq ||z_1| - |z_2||$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, if $z_2 \neq 0$
- $\mathrm{Re}(z) = (z + \bar{z})/2$ and $\mathrm{Im}(z) = (z - \bar{z})/2j$



## Argument of complex numbers




$$x = r\cos\theta, \quad y = r\sin\theta$$
$$z = r(\cos\theta + j\sin\theta)$$
$$r = |z|$$
$$\theta = \tan^{-1}(y/x) \ \triangleq \ \arg z$$
(called an argument of $z$)

**principal value of** $\arg z$ denoted by $\mathrm{Arg}\, z$ is the unique $\theta$ such that $-\pi < \theta \leq \pi$

$$\arg z = \mathrm{Arg}\, z + 2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

**example:** $\mathrm{Arg}(-1 + j) = \frac{3\pi}{4}, \quad \arg z = \frac{3\pi}{4} + 2n\pi, \quad n = 0, \pm 1, \ldots$

## Polar representation

**Euler's formula** ✌

$$e^{j\theta} = \cos\theta + j\sin\theta$$

a polar representation of $z = x + jy$ (where $z \neq 0$) is

$$z = re^{j\theta}$$

where $r = |z|$ and $\theta = \arg z$

**example:**

$$(-1 + j) = \sqrt{2}\, e^{j3\pi/4} = \sqrt{2}\, e^{j(3\pi/4+2n\pi)}, \quad n = 0, \pm 1, \ldots$$

(there are infinite numbers of polar forms for $-1 + j$)



let $z_1 = r_1\, e^{j\theta_1}$ and $z_2 = r_2\, e^{j\theta_2}$

**properties** ✎

- $z_1 z_2 = r_1 r_2\, e^{j(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\, e^{j(\theta_1-\theta_2)}$
- $z^{-1} = \frac{1}{r} e^{-j\theta}$
- $z^n = r^n\, e^{jn\theta}, \quad n = 0, \pm 1, \ldots$

**de Moivre's formula** ✎

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta, \quad n = 0, \pm 1, \pm 2, \ldots$$



**example:** prove the following trigonometric identity

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

from de Moivre's formula,

$$\begin{aligned}\cos 3\theta + j\sin 3\theta &= (\cos\theta + j\sin\theta)^3 \\ &= \cos^3\theta + j3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta\end{aligned}$$

and the identity is readily obtained from comparing the real part of both sides

## Arguments of products

an argument of the product $z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$ is given by

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

**example:** $z_1 = -1$ and $z_2 = -1 + j$

$$\arg(z_1 z_2) = \arg(1-j) = 7\pi/4, \quad \arg z_1 + \arg z_2 = \pi + 3\pi/4$$

this result is not always true if $\arg$ is replaced by $\mathrm{Arg}$

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(1-j) = -\pi/4, \quad \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \pi + 3\pi/4$$



✎ **more properties of the argument function**

- $\arg(\bar{z}) = -\arg z$
- $\arg(1/z) = -\arg z$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(no need to memorize these formulae)



## Roots of complex numbers

an $n$th root of $z_0 = r_0 e^{j\theta_0}$ is a number $z = re^{j\theta}$ such that $z^n = z_0$, or

$$r^n e^{jn\theta} = r_0 e^{j\theta_0}$$

**note:** two nonzero complex numbers

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2k\pi$$

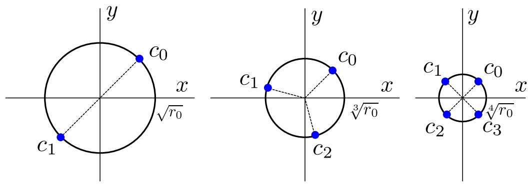
for some $k = 0, \pm 1, \pm 2, \ldots$

therefore, the $n$th roots of $z_0$ are

$$z = \sqrt[n]{r_0}\ \text{exp}\left[j\left(\frac{\theta_0 + 2k\pi}{n}\right)\right] \qquad k = 0, \pm 1, \pm 2, \ldots$$

all of the **distinct** roots are obtained by

$$c_k = \sqrt[n]{r_0}\ \text{exp}\left[j\left(\frac{\theta_0 + 2k\pi}{n}\right)\right] \qquad k = 0, 1, \ldots, n-1$$




the roots lie on the circle $|z| = \sqrt[n]{r_0}$ and equally spaced every $2\pi/n$ rad



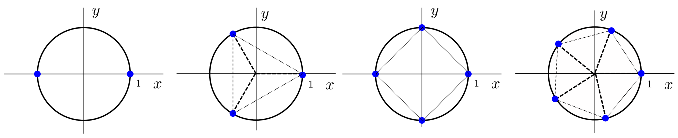
when $-\pi < \theta_0 \le \pi$, we say $c_0$ is the **principal root**

**example 1:** find the $n$ roots of $1$ for $n = 2, 3, 4$ and $5$

$$1 = 1 \cdot \text{exp}\,[j(0 + 2k\pi)], \quad k = 0, \pm 1, \pm 2, \ldots$$

the distinct $n$ roots of $1$ are

$$c_k = \sqrt[n]{r_0}\ \text{exp}\left[j\left(\frac{0 + 2k\pi}{n}\right)\right] \qquad k = 0, 1, \ldots, n-1$$




$n = 2$ $\quad n = 3$ $\quad n = 4$ $\quad n = 5$

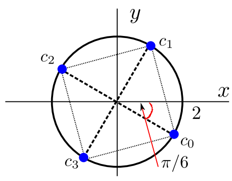


**example 2:** find $(-8 - j8\sqrt{3})^{1/4}$

write $z_0 = -8 - j8\sqrt{3} = 16e^{j(-\pi+\pi/3)} = 16e^{j(-2\pi/3)}$

the four roots of $z_0$ are

$$c_k = (16)^{1/4}\ \text{exp}\left[j\left(\frac{-2\pi/3 + 2k\pi}{4}\right)\right] \qquad k = 0, 1, 2, 3$$



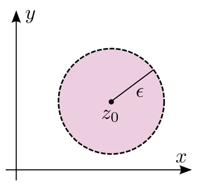

$$\begin{aligned}
c_0 &= 2e^{j(-2\pi/12)} &&= 2e^{-j\pi/6} &&= \sqrt{3} - j \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+2\pi}{4}\right)} &&= 2e^{j\pi/3} &&= 1 + j\sqrt{3} \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+4\pi}{4}\right)} &&= 2e^{j5\pi/6} &&= -\sqrt{3} + j \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+6\pi}{4}\right)} &&= 2e^{j4\pi/3} &&= -1 - j\sqrt{3}
\end{aligned}$$

## Regions in the Complex Plane

- interior, exterior, boundary points
- open and closed sets
- loci on the complex plane



## Regions in the complex plane




an $\epsilon$ **neighborhood** of $z_0$ is the set

$$\{z \in \mathbf{C} \mid |z - z_0| < \epsilon\}$$

**Definition:** a point $z_0$ is said to be

- an **interior point** of a set $S$ if there exists a neighborhood of $z_0$ that contains *only points* of $S$
- an **exterior point** of $S$ when there exists a neighborhood of it containing *no points* of $S$
- a **boundary point** of $S$ if it is neither an interior nor an exterior point of $S$

the **boundary** of $S$ is the set of *all* boundary points of $S$



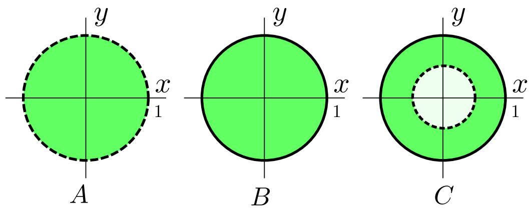
**examples on the real axis:** $S_1 = (0, 1)$, $S_2 = [0, 1]$, and $S_3 = (0, 1]$

in *real analysis*, an $\epsilon$ neighborhood of $x_0 \in \mathbf{R}$ is the set

$$\{x \in \mathbf{R} \mid |x - x_0| < \epsilon\}$$

- any $x \in (0, 1)$ is an interior point of $S_1$, $S_2$, and $S_3$
- any $x \in (-\infty, 0) \cup (1, \infty)$ is an exterior point of $S_1, S_2$ and $S_3$
- $0$ and $1$ are boundary points of $S_1, S_2$ and $S_3$

**examples on the complex plane:**




- any point $z \in \mathbf{C}$ with $|z| < 1$ is an interior point of $A$ and $B$
- any point $z \in \mathbf{C}$ with $1/2 < |z| < 1$ is an interior point of $C$
- any point $z \in \mathbf{C}$ with $|z| > 1$ is an exterior point of $A$ and $B$
- any point $z \in \mathbf{C}$ with $0 < |z| < 1/2$ or $|z| > 1$ is an exterior point of $C$
- the circle $|z| = 1$ is the boundary of $A$ and $B$
- the union of the circles $|z| = 1$ and $|z| = 1/2$ is the boundary of $C$



## Open and Closed sets

- a set is **open** if and only if each of its points is an interior point
- a set is **closed** if it contains all of its boundary points
- the **closure** of a set $S$ is the *closed* set consisting of all points in $S$ together with the boundary of $S$
- some sets are neither open nor closed

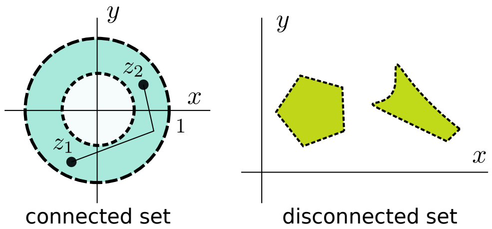
from the examples on page 8-18 and page 8-19,

- $S_1$ is open, $S_2$ is closed, $S_3$ is neither open nor closed
- $S_2$ is the closure of $S_1$
- $A$ is open, $B$ is closed, $C$ is neither open nor closed
- $B$ is the closure of $A$



## Connected sets

an open set $S$ is said to be **connected** if any pair of points $z_1$ and $z_2$ in $S$ can be joined by a *polygonal line* that lies entirely in $S$



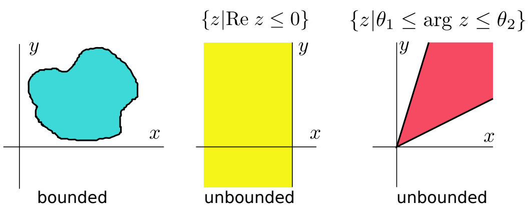

- a nonempty open set that is connected is called a **domain**
- any neighborhood is a domain
- a domain with some, none, or all of its boundary points is called a **region**

## Bounded sets

a set $S$ is said to be **bounded** if for any point $z \in S$,

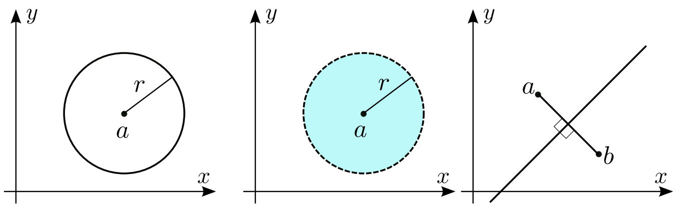
$$|z| \leq M, \quad \text{for some } M < \infty$$

otherwise it is **unbounded**






## Loci in the complex plane




- $|z - a| = r$, $a \in \mathbf{C}, r \in \mathbf{R}$
- $|z - a| < r$, $a \in \mathbf{C}, r \in \mathbf{R}$
- $|z - a| = |z - b|$, $a, b \in \mathbf{C}$



## References

Chapter 1 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

## Exercises

1. Show that
$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\cos\theta\sin\theta$$
using de Moivre's formula.

2. Find all the distinct values of the follow roots:
   (a) the $5$th roots of $-4 + j3$
   (b) the $8$th roots of $\dfrac{1+j}{\sqrt{3}-j}$

3. Show that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$. Show that
$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}).$$
Use this to prove that the triangle inequality $|z+w| \leq |z| + |w|$.

4. Show that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$. Show that
$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}).$$
Use this to prove that the triangle inequality $|z+w| \leq |z| + |w|$.

5. For $n \geq 1$, show that
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n+1/2)\theta]}{2\sin(\theta/2)}.$$
This is known as Lagrange's trigonometric identity.

6. Sketch the following sets and determine which are domains:
   (a) $|z - 3 + j| = 2$
   (b) $|z| = \arg z$
   (c) $|\arg z| < \pi/4$
   (d) $|3z - 2| > 6$
   (e) $\operatorname{Im} z > 2$
   (f) $|z-2| + |z+2| < 6$

# บทที่ 9

# Analytic Functions

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถวิเคราะห์ได้ว่าฟังก์ชันเชิงซ้อนหนึ่งๆ มีอนุพันธ์หรือไม่ และหาอนุพันธ์ที่จุดหนึ่งๆ ได้อย่างไร โดยการใช้นิยามของอนุพันธ์ หรือการประยุกต์จากทฤษฎีบทที่ใช้สมการโคชี-รีมันน์ (Cauchy-Riemann equations)
- สามารถวิเคราะห์ได้ว่าฟังก์ชันเชิงซ้อนหนึ่งๆ เป็นฟังก์ชันวิเคราะห์หรือไม่ รวมถึงสามารถยกตัวอย่างฟังก์ชันวิเคราะห์ที่สำคัญได้

EE202 - EE MATH II Jitkomut Songsiri

# 9. Analytic Functions

- functions of complex variables
- mappings
- limits, continuity, and derivatives
- Cauchy-Riemann equations
- analytic functions

9-1

## Functions of complex variables

a function $f$ defined on a set $S$ is a rule that assigns a complex number $w$ to each $z \in S$

- $S$ is called the **domain** of definition of $f$
- $w$ is called the **value** of $f$ at $z$, denoted by $w = f(z)$
- the domain of $f$ is the set of $z$ such that $f(z)$ is well-defined
- if the value of $f$ is always real, then $f$ is called a **real-valued** function

**example:** $f(z) = 1/|z|$

- let $z = x + jy$ then $f(z) = 1/(x^2 + y^2)$
- $f$ is a real-valued function
- the domain of $f$ is $\mathbf{C}\backslash\{0\}$

Analytic Functions 9-2

suppose $w = u + jv$ is the value of a function $f$ at $z = x + jy$, so that

$$u + jv = f(x + jy)$$

then we can express $f$ in terms of a pair of real-valued functions of $x, y$

$$f(z) = u(x, y) + jv(x, y)$$

**example:** $f(z) = 1/(z^2 + 1)$

- the domain of $f$ is $\mathbf{C}\backslash\{\pm j\}$
- for $z = x + jy$, we can write $f(z) = u(x, y) + jv(x, y)$ by

$$f(x + jy) = \frac{1}{x^2 - y^2 + 1 + j2xy} = \frac{x^2 - y^2 + 1 - j2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$u(x, y) = \frac{x^2 - y^2 + 1}{(x^2 - y^2 + 1)^2 + 4x^2y^2)}, \quad v(x, y) = -\frac{2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2)}$$

Analytic Functions 9-3

if the polar coordinate $r$ and $\theta$ is used, then we can express $f$ as

$$f(re^{j\theta}) = u(r,\theta) + jv(r,\theta)$$

**example:** $f(z) = z + 1/z$, $z \neq 0$

$$\begin{aligned} f(re^{j\theta}) &= re^{j\theta} + (1/r)e^{-j\theta} \\ &= (r + 1/r)\cos\theta + j(r - 1/r)\sin\theta \end{aligned}$$



## Mappings

consider $w = f(z)$ as a *mapping* or a *transformation*

**example:**

- translation each point $z$ by $1$

$$w = f(z) = z + 1 = (x+1) + jy$$

- rotate each point $z$ by $90^\circ$

$$w = f(z) = iz = re^{j(\theta+\pi/2)}$$

- reflect each point $z$ in the real axis

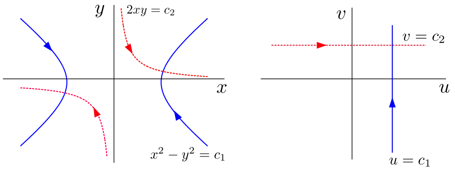
$$w = f(z) = \bar{z} = x - jy$$

it is useful to sketch images under a given mapping



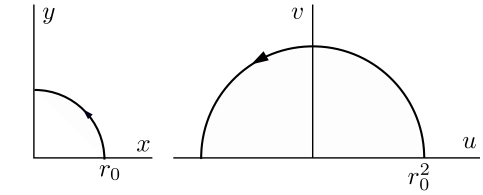
**example 1:** given $w = z^2$, sketch the image of the mapping on the $xy$ plane

$$w = u(x,y) + jv(x,y), \quad \text{where} \quad u = x^2 - y^2, \quad v = 2xy$$




- for $c_1 > 0$, $x^2 - y^2 = c_1$ is mapped onto the line $u = c_1$
- if $u = c_1$ then $v = \pm 2y\sqrt{y^2 + c_1}$, where $-\infty < y < \infty$
- for $c_2 > 0$, $2xy = c_2$ is mapped into the line $v = c_2$
- if $v = c_2$ then $u = c_2^2/4y^2 - y^2$ where $-\infty < y < 0$, or
- if $v = c_2$ then $u = x^2 - c_2^2/4x^2$, $0 < x < \infty$

**example 2:** sketch the mapping $w = z^2$ in the polar coordinate




the mapping $w = r^2 e^{j2\phi} = \rho e^{j\theta}$ where

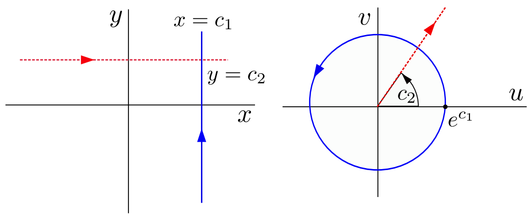
$$\rho = r^2, \quad \theta = 2\phi$$

- the image is found by squaring the modulus and doubling the value $\theta$
- we map the first quadrant onto the upper half plane $\rho \geq 0, 0 \leq \theta \leq \pi$
- we map the upper half plane onto the entire $w$ plane



**mappings by the exponential function:** $w = e^z$

$$w = e^{x+jy} = \rho e^{j\phi}, \quad \text{where} \quad \rho = e^x, \phi = y$$




- a vertical line $x = c_1$ is mapped into the circle of radius $c_1$
- a horizontal line $y = c_2$ is mapped into the ray $\phi = c_2$

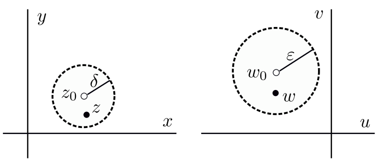


## Limits

**limit** of $f(z)$ as $z$ approaches $z_0$ is a number $w_0$, *i.e.*,

$$\lim_{z \to z_0} f(z) = w_0$$

meaning: $w = f(z)$ can be made arbitrarily close to $w_0$ if $z$ is close enough to $z_0$




**Definition:** if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

then $w_0 = \lim_{z \to z_0} f(z)$

**example:** let $f(z) = 2j\bar{z}$, show that $\lim_{z \to 1} f(z) = 2j$

we must show that for *any* $\varepsilon > 0$, we can *always* find $\delta > 0$ such that

$$|z - 1| < \delta \quad \Longrightarrow \quad |2j\bar{z} - 2j| < \varepsilon$$

if we express $|2j\bar{z} - 2j|$ in terms of $|z - 1|$ by

$$|2j\bar{z} - 2j| = 2|\bar{z} - 1| = 2|z - 1|$$

hence if $\delta = \varepsilon/2$ then

$$|f(z) - 2j| = 2|z - 1| < 2\delta < \varepsilon$$

$f(z)$ can be made arbitrarily close to $2j$ by making $z$ close to $1$ enough

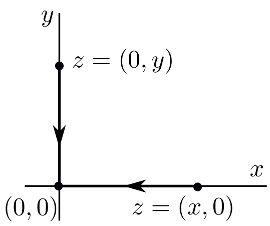
how close ? determined by $\delta$ and $\varepsilon$



**Remarks:**

- when a limit of $f(z)$ exists at $z_0$, it is **unique**
- if the limit exists, $z \to z_0$ means $z$ approaches $z_0$ in any *arbitrary* direction

**example:** let $f(z) = z/\bar{z}$




- if $z = x$ then $f(z) = \frac{x+j0}{x-j0} = 1$
  as $z \to 0$, $f(z) \to 1$ along the real axis
- if $z = jy$ then $f(z) = \frac{0+jy}{0-jy} = -1$
  as $z \to 0$, $f(z) \to -1$ along the imaginary axis

since a limit must be unique, we conclude that $\lim_{z \to 0} f(z)$ *does not* exist



## Theorems on limits

**Theorem** ✌ suppose $f(z) = u(x, y) + jv(x, y)$ and

$$z_0 = x_0 + jy_0, \quad w_0 = u_0 + jv_0$$

then $\lim_{z \to z_0} f(z) = w_0$ *if and only if*

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$$

**Theorem** ✌ suppose $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = c_0$ then

- $\displaystyle\lim_{z \to z_0} [f(z) + g(z)] = w_0 + c_0$
- $\displaystyle\lim_{z \to z_0} [f(z)g(z)] = w_0 c_0$
- $\displaystyle\lim_{z \to z_0} \frac{f(z)}{g(z)} = w_0/c_0$ if $c_0 \neq 0$

**Limit of polynomial functions:** for $p(z) = a_0 + a_1 z + \cdots + a_n z^n$

$$\lim_{z \to z_0} p(z) = p(z_0)$$

**Theorem** ✎ suppose $\lim_{z \to z_0} f(z) = w_0$ then

- $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$

**example:**

$$\lim_{z \to \infty} \frac{2z + j}{z + 1} = 2 \quad \text{because} \quad \lim_{z \to 0} \frac{(2/z) + j}{(1/z) + j} = \lim_{z \to 0} \frac{2 + jz}{1 + z} = 2$$



## Continuity

**Definition:** $f$ is said to be **continuous** at a point $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

provided that both terms must exist

this statement is equivalent to another definition:

$\delta - \varepsilon$ **Definition:** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

then $f$ is continuous at $z_0$



**example:** $f(z) = z/(z^2 + 1)$

- $f$ is not continuous at $\pm j$ because $f(\pm j)$ do not exist
- $f$ is continuous at $1$ because

$$f(1) = 1/2 \quad \text{and} \quad \lim_{z \to 1} \frac{z}{z^2 + 1} = 1/2$$

**example:** $f(z) = \begin{cases} \frac{z^2 + j3z - 2}{z + j}, & z \neq -j \\ 2j, & z = -j \end{cases}$

$$\lim_{z \to -j} f(z) = \lim_{z \to -j} \frac{z^2 + j3z - 2}{z + j} = \lim_{z \to -j} \frac{(z + j)(z + j2)}{z + j} = \lim_{z \to -j} (z + j2) = j$$

we see that $\lim_{z \to -j} f(z) \neq f(-j) = 2j$

hence, $f$ is not continuous at $z = -j$

**Remarks** ✎

- $f$ is said to be continuous in a region $R$ if it is continuous at *each point* in $R$
- if $f$ and $g$ are continuous at a point, then so is $f+g$
- if $f$ and $g$ are continuous at a point, then so is $fg$
- if $f$ and $g$ are continuous at a point, then so is $f/g$ at any such point if $g$ is not zero there
- if $f$ and $g$ are continuous at a point, then so is $f \circ g$
- $f(z) = u(x,y) + jv(x,y)$ is continuous at $z_0 = (x_0, y_0)$ if and only if

$$u(x,y) \text{ and } v(x,y) \text{ are continuous at } (x_0, y_0)$$

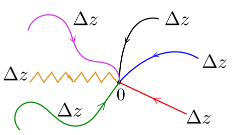


## Derivatives

the **complex derivative** of $f$ at $z$ is the limit

$$\frac{df}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

(if the limit exists)




$\Delta z$ is a complex variable

so the limit must be the same no matter how $\Delta z$ approaches $0$

$f$ is said to be **differentiable** at $z$ when $f'(z)$ exists



**example:** find the derivative of $f(z) = z^3$

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{3z^2 \Delta z + 3z\Delta z^2 + \Delta z^3}{\Delta z} \\
&= \lim_{\Delta z \to 0} 3z^2 + 3z\Delta z + \Delta z^2 = 3z^2
\end{aligned}$$

hence, $f$ is differentiable at any point $z$ and $f'(z) = 3z^2$

**example:** find the derivative of $f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

but $\lim_{z \to 0} z/\bar{z}$ does not exist (page 9-11), so $f$ is not differentiable everywhere

**example:** $f(z) = |z|^2$ (real-valued function)

$$\begin{aligned}\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - |z|^2}{\Delta z} \\ &= \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \\ &= \begin{cases} \bar{z} + \Delta z + z, & \Delta z = \Delta x + j0 \\ \bar{z} - \Delta z - z, & \Delta z = 0 + j\Delta y \end{cases}\end{aligned}$$

hence, if $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists then it must be unique, meaning

$$\bar{z} + z = \bar{z} - z \quad \Longrightarrow \quad z = 0$$

therefore $f$ is only differentiable at $z = 0$ and $f'(0) = 0$



**note:** $f(z) = |z|^2 = u(x,y) + jv(x,y)$ where

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0$$

- $f$ is continuous everywhere because $u(x,y)$ and $v(x,y)$ are continuous
- but $f$ is *not* differentiable everywhere; $f'$ only exists at $z = 0$

hence, for any $f$ we can conclude that

- the continuity of a function *does not* imply the existence of a derivative !
- however, the existence of a derivative *implies* the continuity of $f$ at that point

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0$$

**Theorem** ✌ if $f(z)$ is differentiable at $z_0$ then $f(z)$ is continuous at $z_0$



## Differentiation formulas

basic formulas still hold for complex-valued functions

- $\frac{dc}{dz} = 0$ and $\frac{d}{dz}[cf(z)] = cf'(z)$ where $c$ is a constant
- $\frac{d}{dz} z^n = nz^{n-1}$ if $n \neq 0$ is an integer
- $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
- $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$ (product rule)
- let $h(z) = g(f(z))$ (chain rule)

$$h'(z) = g'(f(z))f'(z)$$

## Cauchy-Riemann equations

✌ **Theorem:** suppose that


$$f(z) = u(x, y) + jv(x, y)$$

and $f'(z)$ exists at $z_0 = (x_0, y_0)$ then

- the first-order derivatives of $u$ and $v$ must exist at $(x_0, y_0)$
- the derivatives must satisfiy the **Cauchy-Riemann equations:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x_0, y_0)$$

and $f'(z_0)$ can be written as

$$f'(z_0) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \qquad \text{(evaluated at } (x_0, y_0))$$

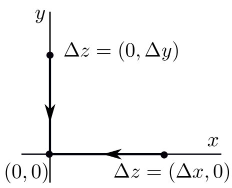


**Proof:** we start by writing

$$z = x + jy, \quad \Delta z = \Delta x + j\Delta y$$

and $\Delta w = f(z + \Delta z) - f(z)$ which is

$$\Delta w = u(x + \Delta x, y + \Delta y) - u(x, y) + j[v(x + \Delta x, y + \Delta y) - v(x, y)]$$




- let $\Delta z \to 0$ horizontally $\qquad (\Delta y = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x + \Delta x, y) - u(x, y) + j[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$

- let $\Delta z \to 0$ vertically $\qquad (\Delta x = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x, y + \Delta y) - u(x, y) + j[v(x, y + \Delta y) - v(x, y)]}{j\Delta y}$$



we calculate $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ in both directions

- as $\Delta z \to 0$ horizontally

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + j\frac{\partial v}{\partial x}(x, y)$$

- as $\Delta z \to 0$ vertically

$$f'(z) = \frac{\partial v}{\partial y}(x, y) - j\frac{\partial u}{\partial y}(x, y)$$

$f'(z)$ must be valid as $\Delta z \to 0$ in any direction

the proof follows by matching the real/imaginary parts of the two expressions

**note:** C-R eqs provide **necessary** conditions for the existence of $f'(z)$

**example:** $f(z) = |z|^2$, we have

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

if the Cauchy-Riemann eqs are to hold at a point $(x, y)$, it follows that

$$2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and

$$2y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

hence, a *necessary condition* for $f$ to be differentiable at $z$ is

$$z = x + jy = 0$$

(if $z \neq 0$ then $f$ is not differentiable at $z$)



## Cauchy-Riemann equations in Polar form

let $z = x + jy = re^{j\theta} \neq 0$ with $x = r\cos\theta$ and $y = r\sin\theta$

apply the Chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x} \cdot r\sin\theta + \frac{\partial u}{\partial y} \cdot r\cos\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x} \cdot r\sin\theta + \frac{\partial v}{\partial y} \cdot r\cos\theta$$

substitute $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (Cauchy-Riemanns equations)

the Cauchy-Riemann equations in the polar form are

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$



**example:** Cauchy-Riemann eqs are satisfied but $f'$ does not exist at $z = 0$

$$f(z) = \begin{cases} \bar{z}^2/z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

from a direct calculation, express $f$ as $f = u(x, y) + jv(x, y)$ where

$$u(x, y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}, \qquad v(x, y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2 + y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}$$

and we can say that

$$u(x, 0) = x, \ \forall x, \quad u(0, y) = 0, \ \forall y, \quad v(x, 0) = 0, \ \forall x, \quad v(0, y) = y, \ \forall y$$

which give

$$\frac{\partial u(x, 0)}{\partial x} = 1, \ \forall x, \quad \frac{\partial u(0, y)}{\partial y} = 0, \ \forall y, \quad \frac{\partial v(x, 0)}{\partial x} = 0, \ \forall x, \quad \frac{\partial v(0, y)}{\partial y} = 1, \ \forall y$$

so the Cauchy-Riemann equations are satisfied at $(x, y) = (0, 0)$

however, $f$ is **not** differentiable at $0$ because

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{(\overline{\Delta z})^2}{\Delta z} - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

and the limit does not exist (from page 9-11)



## Sufficient conditions for differentiability

☃ **Theorem:** let $z = x + jy$ and let the function

$$f(z) = u(x, y) + jv(x, y)$$

be defined on some neighborhood of $z$, and suppose that

1. the first partial derivatives of $u$ and $v$ w.r.t. $x$ and $y$ exist
2. the partial derivatives are **continuous** at $(x, y)$ and satisfy **C-R eqs**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x, y)$$

then $f'(z)$ exists and its value is

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + j\frac{\partial v}{\partial x}(x, y)$$



**example 1:** on page 9-27, $f'(0)$ does not exist while the C-R eqs hold because

$$\frac{\partial u(x, y)}{\partial x} = \frac{x^4 - 3y^4 + 6x^2y^2}{(x^2 + y^2)^2} \quad \Longrightarrow \quad \frac{\partial u(x, 0)}{\partial x} = 1, \quad \frac{\partial u(0, y)}{\partial x} = -3$$

which show that $\frac{\partial u}{\partial x}$ is *not continuous* at $(x, y) = (0, 0)$ (neither is $\frac{\partial v}{\partial y}$ )

**example 2:** $f(z) = z^2 = x^2 - y^2 + j2xy$, find $f'(z)$ if it exists

check the Cauchy-Riemann eqs,

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

and all the partial derivatives are continuous at $(x, y)$

thus, $f'(z)$ exists and

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = 2x + j2y = 2z$$

**example 3:** $f(z) = e^z$, find $f'(z)$ if it exists

write $f(z) = e^x \cos y + je^x \sin y$

check the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$$

and all the derivatives are continuous for all $(x, y)$

thus $f'(z)$ exists everywhere and

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = e^x \cos y + je^x \sin y$$

note that $f'(z) = e^z = f(z)$ for all $z$



## Analytic functions

**Definition:** $f$ is said to be **analytic** at $z_0$ if it has a derivative at $z_0$ and every point in some neighborhood of $z_0$

- the terms **regular** and **holomorphic** are also used to denote analyticity
- we say $f$ is analytic on a domain $D$ if it has a derivative *everywhere* in $D$
- if $f$ is analytic at $z_0$ then $z_0$ is called a **regular** point of $f$
- if $f$ is not analytic at $z_0$ but is analytic at some point in every neighborhood of $z_0$ then $z_0$ is called a **singular** point of $f$
- a function that is analytic at *every point* in the complex plane is called **entire**



let $f(z) = u(x, y) + jv(x, y)$ be defined on a domain $D$

✌ **Theorem:** $f(z)$ is analytic on $D$ *if and only if* all of followings hold

- $u(x, y)$ and $v(x, y)$ have *continuous* first-order partial derivatives
- the Cauchy-Riemann equations are satisfied

**examples** ✎

- $f(z) = z$ is analytic everywhere ($f$ is entire)
- $f(z) = \bar{z}$ is not analytic everywhere because

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

**more examples** ✎

- $f(z) = e^z = e^x \cos x + je^x \sin y$ is analytic everywhere ($f$ is entire)

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

  and all the partial derivatives are continuous
- $f(z) = (z+1)(z^2+1)$ is analytic on **C** ($f$ is entire)
- $f(z) = \dfrac{(z^3+1)}{(z^2-1)(z^2+4)}$ is analytic on **C** except at

$$z = \pm 1, \quad \text{and} \quad z = \pm j2$$

- $f(z) = xy + jy$ is not analytic everywhere because

$$\frac{\partial u}{\partial x} = y \neq 1 = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = x \neq 0 = -\frac{\partial v}{\partial x}$$



## Theorem on analytic functions

let $f$ be an analytic function everywhere in a domain $D$

**Theorem:** if $f'(z) = 0$ everywhere in $D$ then $f(z)$ must be constant on $D$

**Theorem:** if $f(z)$ is real valued for all $z \in D$ then $f(z)$ must be constant on $D$



## Harmonic functions

the equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

is called **Laplace's equation**

we say a function $u(x,y)$ is **harmonic** if

- the first- and second-order partial derivatives exist and are continuous
- $u(x,y)$ satisfy Laplace's equation

✎ **Theorem:** if $f(z) = u(x,y) + jv(x,y)$ is analytic in a domain $D$ then $u$ and $v$ are harmonic in $D$

**example:** $f(z) = e^{-y}\sin x - je^{-y}\cos x$

- $f$ is entire because

$$\frac{\partial u}{\partial x} = e^{-y}\cos x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^{-y}\sin x = -\frac{\partial v}{\partial x}$$

(C-R is satisfied for every $(x, y)$ and the partial derivatives are continuous)

- we can verify that

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^{-y}\cos x, & \frac{\partial^2 u}{\partial x^2} &= -e^{-y}\sin x \\ \frac{\partial u}{\partial y} &= -e^{-y}\sin x, & \frac{\partial^2 u}{\partial y^2} &= e^{-y}\sin x \end{aligned}\right\} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- hence, $u(x, y) = e^{-y}\sin x$ is harmonic in every domain of the complex plane



## Harmonic Conjugate

$v$ is said to be a **harmonic conjugate** of $u$ if

1. $u$ and $v$ are harmonic in a domain $D$
2. their first-order partial derivatives satisfy the Cauchy-Riemann equations on $D$

**example:** $f(z) = z^2 = x^2 - y^2 + j2xy$

- since $f$ is entire, then $u$ and $v$ are harmonic on the complex plane
- since $f$ is analytic, $u$ and $v$ satisfy the C-R equations
- therefore, $v$ is a harmonic conjugate of $u$



✌ **Theorem:** $f(z) = u(x, y) + jv(x, y)$ is analytic in a domain $D$ *if and only if*

$v$ is a harmonic conjugate of $u$

**example:** $f = 2xy + j(x^2 - y^2)$

- $f$ is not analytic anywhere because

$$\frac{\partial u}{\partial x} = 2y \neq -2y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = 2x \neq -2x = -\frac{\partial v}{\partial x}$$

(C-R eqs do not hold anywhere except $z = 0$)

- hence, $x^2 - y^2$ cannot be a harmonic conjugate of $2xy$ on any domain

(contrary to the example on page 9-38)

## References

Chapter 2 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 2 in

T. W. Gamelin, *Complex Analysis*, Springer, 2001

## Exercises

1. Let $n$ be a positive integer. Use the $\delta - \epsilon$ definition of a limit to prove that

$$\lim_{z \to 0} z^n = 0.$$

2. Determine whether $f(z)$ is continous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function $f$ is equal to:
   (a) $|z|^2 \operatorname{Im}(1/z)$
   (b) $(\operatorname{Re} z^2)/|z|$
   (c) $(\operatorname{Im} z^2)/|z|^2$
3. Prove that $f(z) = z^n$ where $n$ is a positive integer, is analytic everywhere, and $f'(z) = nz^{n-1}$.
4. Where is the function $f(z) = z \operatorname{Re} z$ differentiable ?
5. Prove that $f(z) = |z|^4$ is differentiable but not analytic at $z = 0$.
6. For each of the following functions, locate the singularities in the finite $z$ plane.
   (a) $\dfrac{z^2 - 3z}{z^2 + 4z + 4}$
   (b) $\sin^{-1}(1/z)$
   (c) $\dfrac{\cos z}{(z + 2j)^2}$
7. Show that the function $u = 2x(1-y)$ is harmonic. Find a function $v$ such that $f(z) = u(x, y) + jv(x, y)$ is analytic, i.e., the conjugate function of $u$.

# บทที่ 10

# Elementary Functions

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ การรู้จักนิยามและคุณสมบัติของฟังก์ชันเชิงซ้อนพื้นฐาน รวมถึงการหาค่าของฟังก์ชันพื้นฐาน ดังต่อไปนี้

- ฟังก์ชันเลขชี้กำลัง
- ฟังก์ชันลอการิทึม
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- ฟังก์ชันไฮเพอร์โบลิก

EE202 - EE MATH II Jitkomut Songsiri

# 10. Elementary Functions

- exponential function
- logarithmic function
- complex components
- trigonometric function
- hyperbolic functions
- branches for multi-valued functions



## Exponential function

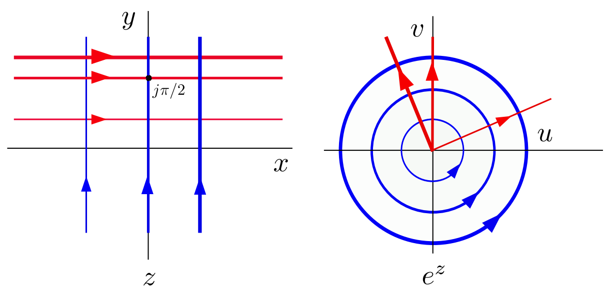
from $z = x + jy$, an exponential function is defined as

$$f(z) = e^z \quad = \quad e^x \cos y + je^x \sin y$$

**Properties** ✎

- $f(z) = e^z$ is analytic everywhere on **C** ($e^z$ is entire)
- $f'(z) = e^z$
- $e^{z+w} = e^z e^w$ , $z, w \in \mathbf{C}$ (addition formula)
- $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$ (so $e^z \neq 0$)
- if $z$ is pure imaginary, then $e^z$ is periodic






images under $f(z) = e^z$

- images of horizontal lines are rays pointing from the origin
- images of vertical lines are circles centered at the origin

## Logarithmic function

the definition of $\log$ function is based on solving

$$e^w = z$$

for $w$ where $z$ is any **nonzero** complex number, and we call $w = \log z$

write $z = re^{j\Theta}(-\pi < \Theta \leq \pi)$ and $w = u + jv$, so we have

$$e^u = r, \qquad v = \Theta + 2n\pi$$

thus the definition of the (multiple-valued) **logarithmic** function of $z$ is

$$\log z = \log r + i(\Theta + 2n\pi), \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

if only the principle value of $\arg z$ is used $(n = 0)$, then $\log z$ is *single-valued*



the **principal value** of $\log z$ is defined as

$$\mathrm{Log}\, z = \log r + i\, \mathrm{Arg}\, z$$

where $r = |z|$ and recall that $\mathrm{Arg}\, z$ is the principal argument of $z$

- $\mathrm{Log}\, z$ is single-valued
- $\log z = \mathrm{Log}\, z + j2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots)$

note: when $z$ is complex, one should **not** jump into the conclusion that

$$\log(e^z) = z \qquad (\log \text{ is multiple-valued})$$

instead, if $z = x + jy$, we should write

$$\begin{aligned}\log(e^z) &= \log|e^z| + j(\mathrm{Arg}(e^z) + 2n\pi) = \log|e^x| + j(y + 2n\pi) \\ &= z + j2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)\end{aligned}$$



**example:** find $\log z$ for $z = -1 + j$, $z = 1$, and $z = -1$

- if $z = -1 + j$ then $r = \sqrt{2}$ and $\mathrm{Arg}\, z = 3\pi/4$

$$\begin{aligned}\log z &= \log\sqrt{2} + j(3\pi/4 + 2n\pi), \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \mathrm{Log}\, z &= \log\sqrt{2} + j3\pi/4\end{aligned}$$

- if $z = 1$ then $r = 1$ and $\mathrm{Arg}\, z = 0$

$$\begin{aligned}\log z &= 0 + j2n\pi = j2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \mathrm{Log}\, z &= 0 \qquad \text{(as expected)}\end{aligned}$$

- if $z = -1$ then $r = 1$ and $\mathrm{Arg}\, z = \pi$

$$\begin{aligned}\log z &= \log 1 + j(\pi + 2n\pi) = j(2n + 1)\pi, \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \mathrm{Log}\, z &= j\pi = j \qquad \text{(now we can find } \log \text{ of a negative number)}\end{aligned}$$

## Complex exponents

let $z \neq 0$ and $c$ be any complex number, the function $z^c$ is defined via

$$z^c = e^{c \log z}$$

where $\log z$ is the multiple-valued logarithmic function

let $\Theta$ be the principal value of $\arg z$ and let $c = a + jb$

$$z^c = e^{c \log z} = e^{(a+jb)(\log |z| + j(\Theta + 2n\pi))}, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

**example:** find $j^j$

$$j^j = e^{j(\log j)} = e^{j(\log 1 + j(\pi/2 + 2n\pi))} = e^{-(1/2+2n)\pi}, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

(complex power of a complex can become real numbers)

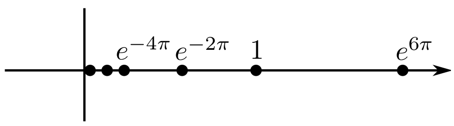


more example: the values of $j^{-j}$ are given by

$$j^{-j} = e^{-j(\log j)} = e^{-j(\log 1 + j(\pi/2 + 2m\pi))} = e^{(1/2+2m)\pi}, \quad (m = 0, \pm 1, \pm 2, \ldots)$$

if we multiply the values of $j^j$ by those of $j^{-j}$ we obtain infinitely many values of

$$e^{2k\pi}, \quad -\infty < k < \infty$$




thus, the usual algebraic rules *do not* apply to $z^c$ when they are multi-valued !

$$(j^j) \cdot (j^{-j}) \neq j^0 = 1$$



## Trigonometric functions

by using Euler's formula

$$e^{jx} = \cos x + j \sin x, \qquad \text{for any } \textit{real} \text{ number } x$$

we can write

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}, \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

hence, it is *natural* to define trigonometric functions of a complex number $z$ as

$$\begin{aligned} \sin z &= \frac{e^{jz} - e^{-jz}}{2j}, & \cos z &= \frac{e^{jz} + e^{-jz}}{2}, & \tan z &= \frac{\sin z}{\cos z} \\ \csc z &= \frac{1}{\sin z}, & \sec z &= \frac{1}{\cos z}, & \cot z &= \frac{1}{\tan z} \end{aligned}$$

**Properties** ✎

- $\sin z$ and $\cos z$ are entire functions (since $e^{jz}$ and $e^{-jz}$ are entire)
- $\dfrac{d}{dz}\sin z = \cos z$ and $\dfrac{d}{dz}\cos z = -\sin z$ (use $\frac{d}{dz}e^{jz} = je^{jz}$)
- $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$ (sine is odd and cosine is even)
- $\sin(z + 2\pi) = \sin z$ and $\sin(z + \pi) = -\sin z$
- $\cos(z + 2\pi) = \cos z$ and $\cos(z + \pi) = -\cos z$
- $\sin(z + \pi/2) = \cos z$ and $\sin(z - \pi/2) = -\cos z$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$



## Hyperbolic functions

the hyperbolic sine, cosine, and tangent of a complex number are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}, \qquad \tanh z = \frac{\sinh z}{\cosh z}$$

(as they are with a real variable)

**Properties** ✎

- $\sinh z$ and $\cosh z$ are entire (since $e^z$ and $e^{-z}$ are entire)
- $\tanh z$ is analytic in every domain in which $\cosh z \neq 0$
- $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$
- $\frac{d}{dz}\tanh z = \operatorname{sech}^2 z$



## Branches for multiple-valued functions

we often need to investigate the differentiability of a function $f$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

what happen if $f$ is multiple-valued (like $\arg z$ , $z^c$) ?

- have to make sure if the two funtion values tend to the same value in the limit
- have to choose one of the function values in a consistent way

restricting the values of a multiple-valued functions to make it single-valued in some region is called choosing **a branch** of the function

a **branch** of $f$ is any single-valued function $F$ that is analytic in some domain

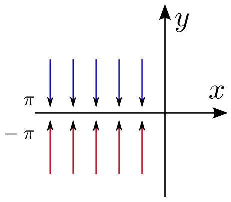
## Branches for logarithmic functions

we define the **principal branch** $\mathrm{Log}$ of the lograrithmic function as

$$\mathrm{Log}\, z = \mathrm{Log}\,|z| + j\,\mathrm{Arg}(z), \quad -\pi < \mathrm{Arg}(z) < \pi$$

where $\mathrm{Arg}(z)$ is the principle value of $\arg(z)$

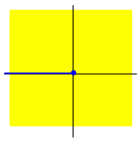
- $\mathrm{Log}\, z$ is single-valued
- $\mathrm{Log}\, z$ is *not continuous* along **the negative real axis** (because of $\mathrm{Arg}\, z$)




$$z = -x + j\epsilon \implies \mathrm{Arg}\, z \to \pi \quad \text{as } \epsilon \to 0$$
$$z = -x - j\epsilon \implies \mathrm{Arg}\, z \to -\pi \quad \text{as } \epsilon \to 0$$



a **branch cut** is portion of curve that is introduced to define a branch $F$



$D = \mathbf{C}\backslash(-\infty, 0]$

- points on the branch cut for $f$ are singular points
- the negative real axis is a **branch cut** for the $\mathrm{Log}$ function
- $\mathrm{Log}\, z$ is single-valued and continuous in $D = \mathbf{C}\backslash(-\infty, 0]$

let $z_0$ be any point in $D$ and $w_0 = \mathrm{Log}\, z_0$ (or $z_0 = e^{w_0}$)

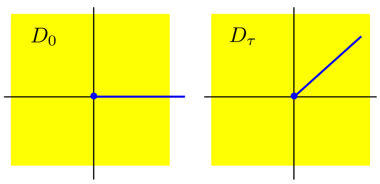
$$\begin{aligned}\frac{d}{dz}\mathrm{Log}\, z_0 &= \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{z \to z_0} \frac{1}{\frac{z - z_0}{w - w_0}} = \lim_{w \to w_0} \frac{1}{\frac{z - z_0}{w - w_0}} \\ &= \lim_{w \to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} = \frac{1}{\frac{d}{dz}e^w\big|_{w = w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}\end{aligned}$$

(we have used single-valuedness and continuity of the Log function)

$\mathrm{Log}\, z$ is analytic in $D$



**other branches of** $\log z$




$$\begin{aligned}\log(z) &= \mathrm{Log}\,|z| + j\arg(z), \quad 0 < \arg(z) < 2\pi \\ \log(z) &= \mathrm{Log}\,|z| + j\arg_\tau(z), \quad \tau < \arg_\tau(z) < \tau + 2\pi\end{aligned}$$

a branch $\log(z)$ is analytic everywhere on $D_\tau$

any point that is common to all branch cuts of $f$ is called a **branch point**

the origin is a **branch point** of the $\log$ function

**example:** suppose we have to compute the derivative of

$$f(z) = \log(z^2 - 1) \quad \text{at point } z = j$$

choose a branch of $f$ which is analytic in a region containing the point

$$j^2 - 1 = -2$$

the principal branch is not analytic there, so we choose another branch

$e.g.$, choose $\log(z) = \operatorname{Log}|z| + j\arg(z)$, $0 < \arg z < 2\pi$; then by chain rule,

$$f'(j) = \left.\frac{2z}{z^2 - 1}\right|_{z=j} = \frac{2j}{j^2 - 1} = -j$$



## Branches for the complex power

define the **principal branch** of $z^c$ to be

$$e^{c \operatorname{Log} z}$$

where $\operatorname{Log} z$ is the principal branch of $\log z$

- since the exponential function is entire, the principal branch of $z^c$ is analytic in $D$ where $\operatorname{Log} z$ is analytic
- using the chain rule

$$\frac{d}{dz}\left.\left(e^{c \operatorname{Log} z}\right)\right|_{z=z_0} = e^{c \operatorname{Log} z_0}\frac{c}{z_0} = \frac{cz_0^c}{z_0} = cz_0^{c-1}$$

  provided that we use the same branch of $z^c$ on both sides of the equation

($z^a z^b = z^{a+b}$ iff we use the same branch for the complex power on both sides)



## References

Chapter 3 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 1 in

T. W. Gameline, *Complex Analysis*, Springer, 2001

J. F. O'Farrill, *Lecture note on Complex Analysis*, University of Edinburgh, `http://www.maths.ed.ac.uk/ jmf/Teaching/MT3/ComplexAnalysis.pdf`

## Exercises

1. Find all values of $z$ such that $e^{4z-1} = 2 + j2\sqrt{3}$.

2. Show that $e^{\bar{z}} = \overline{e^{z}}$

3. What restriction is placed on $z$ ? when

   (a) $e^z$ is real;

   (b) $e^z$ is pure imaginary.

4. Find all values of $\log z$ and the principle value $\operatorname{Log} z$ when $z$ equals

$$-je^2, \quad 2 + j2, \quad 4 - j3.$$

5. Show that the set of values of $\log(j^3)$ is not the same as the set of values of $3 \log j$.

6. Find $j^{j^j}$. Show that it does not coincide with $j^{j \cdot j} = j^{-1}$.

7. Find the principle values of the following complex numbers.

$$(-3)^{3-j}, \quad (1 - j)^{1+j}, \quad (-1)^{2-j}$$

8. Determine all the values of (a) $(1 + j)^j$ (b) $1^{\sqrt{2}}$.

9. Establish the following addition formulae:

   (a) $\cos(z + w) = \cos z \cos w - \sin z \sin w$

   (b) $\sin(z + w) = \sin z \cos w + \cos z \sin w$

   (c) $\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w$

   (d) $\sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w$

10. Show that

$$\tan^{-1} z = \frac{1}{j2} \log\left(\frac{1 + jz}{1 - jz}\right),$$

where both sides of the identify are to be interpreted as subsets of the complex plane, i.e., $\tan^{-1} z$ is a multiple-valued function. In other words, show that $\tan w = z$ if and only if $j2w$ is one of the logarithm values on the right hand side.

# บทที่ 11

# Integrals

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถหาค่าอินทิกรัลตามเส้นรอบขอบ (contour integral) และวิเคราะห์ได้ว่าอินทิกรัลดังกล่าวขึ้นกับทางเดิน (path) หรือไม่
- สามารถหาค่าอินทิกรัลตามเส้นรอบขอบปิด (closed contour integral) ด้วยการประยุกต์ใช้ทฤษฎีบทของโคชีได้

# 11. Integrals

- derivatives of functions
- definite integrals
- contour integrals
- Cauchy-Goursat theorem
- Cauchy integral formula



## Derivatives of functions

consider derivatives of complex-valued functions $w$ of a *real* variable $t$

$$w(t) = u(t) + jv(t)$$

where $u$ and $v$ are **real-valued** functions of $t$

the **derivative** $w'(t)$ or $\frac{d}{dt}w(t)$ is defined as

$$w'(t) = u'(t) + jv'(t)$$

**Properties** ✎ many rules are carried over to complex-valued functions

- $[cw(t)]' = cw'(t)$
- $[w(t) + s(t)]' = w'(t) + s'(t)$
- $[w(t)s(t)]' = w'(t)s(t) + w(t)s'(t)$



**mean-value theorem:** no longer applies for complex-valued functions

suppose $w(t)$ is continuous on $[a, b]$ and $w'(t)$ exists

it is *not necessarily true* that there is a number $c \in [a, b]$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for example, $w(t) = e^{jt}$ on the interval $[0, 2\pi]$ and we have $w(2\pi) - w(0) = 0$

however, $|w'(t)| = |je^{jt}| = 1$, which is never zero

## Definite integrals

the **definite integral** of a complex-valued function

$$w(t) = u(t) + jv(t)$$

over an interval $a \leq t \leq b$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + j\int_a^b v(t)dt$$

provided that each integral exists (ensured if $u$ and $v$ are piecewise continuous)

**Properties** ✎

- $\int_a^b [cw(t) + s(t)]dt = c\int_a^b w(t)dt + \int_a^b s(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$



**Fundamental Theorem of Calculus:** still holds for complex-valued functions

suppose

$$W(t) = U(t) + jV(t) \quad \text{and} \quad w(t) = u(t) + jv(t)$$

are *continuous* on $[a, b]$

if $W'(t) = w(t)$ when $a \leq t \leq b$ then $U'(t) = u(t)$ and $V'(t) = v(t)$

then the integral becomes

$$\int_a^b w(t)dt = U(t)|_a^b + j\,V(t)|_a^b = [U(b) + jV(b)] - [U(a) + jV(a)]$$

therefore, we obtain

$$\int_a^b w(t)dt = W(b) - W(a)$$



**example:** compute $\int_0^{\pi/6} e^{j2t}dt$

since

$$\frac{d}{dt}\left(\frac{e^{j2t}}{j2}\right) = e^{j2t}$$

the integral is given by

$$\begin{aligned} \int_0^{\pi/6} e^{j2t}dt &= \frac{1}{j2}e^{j2t}\Big|_0^{\pi/6} \\ &= \frac{1}{j2}[e^{j\pi/3} - e^{j0}] \\ &= \frac{\sqrt{3}}{4} + \frac{j}{4} \end{aligned}$$

**mean-value theorem for integration:** not hold for complex-valued $w(t)$

it is *not necessarily true* that there exists $c \in [a, b]$ such that

$$\int_a^b w(t)dt = w(c)(b-a)$$

for example, $w(t) = e^{jt}$ for $0 \le t \le 2\pi$ (same example as on page 11-3)

it is easy to see that

$$\int_a^b w(t)dt = \int_0^{2\pi} e^{jt}dt = \left.\frac{e^{jt}}{j}\right|_0^{2\pi} = 0$$

but there is no $c \in [0, 2\pi]$ such that $w(c) = 0$



## Contour integral

integrals of complex-valued functions defined on **curves** in the complex plane

- arcs
- contours
- contour integrals



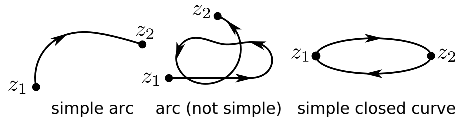
## Arcs

a set of points $z = (x, y)$ in the complex plane is said to be an **arc** or a **path** if

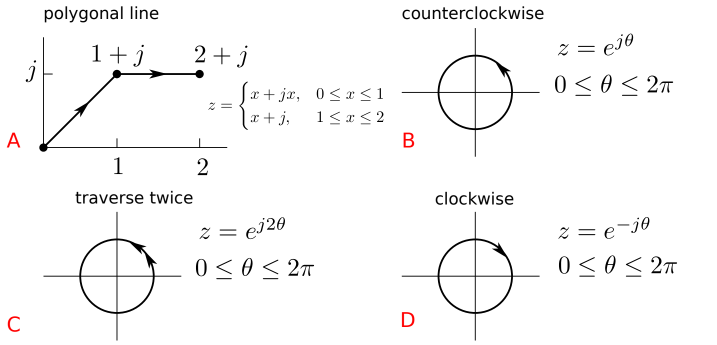
$$x = x(t), \quad y = y(t), \text{ or } z(t) = x(t) + jy(t), \qquad a \le t \le b$$

where $x(t)$ and $y(t)$ are *continuous* functions of real parameter $t$

- the arc is **simple** or is called a **Jordan arc** if it does not cross itself, $e.g.$, $z(t) \ne z(s)$ when $t \ne s$
- the arc is **closed** if it starts and ends at the same point, $e.g.$, $z(b) = z(a)$
- a **simple closed path (or curve)** is a *closed* path such that $z(t) \ne z(s)$ for $a \le s < t < b$



**examples:**




the arcs $B, C$ and $D$ have the same set of points, but they are *not* the same arc

**remark:** a closed curve is **positive oriented** if it is counterclockwise direction



## Contours

an arc is called **differentiable** if the components $x'(t)$ and $y'(t)$ of the derivative

$$z'(t) = x'(t) + jy'(t)$$

of $z(t)$ used to represent the arc, are **continuous** on the interval $[a, b]$

the arc $z = z(t)$ for $a \leq t \leq b$ is said to be **smooth** if

- $z'(t)$ is continuous on the closed interval $[a, b]$
- $z'(t) \neq 0$ throughout the open interval $a < t < b$

a concatenation of smooth arcs is called a **contour** or **piecewise smooth arc**



## Contour integrals

let $C$ be a contour extending from a point $a$ to a point $b$

an integral defined in terms of the values $f(z)$ along a contour $C$ is denoted by

- $\int_C f(z)dz$ (its value, in general, depends on $C$)
- $\int_a^b f(z)dz$ (if the integral is *independent* of the choice of $C$)

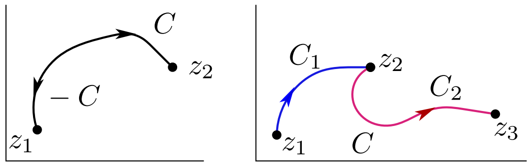
if we assume that $f$ is **piecewise continuous** on $C$ then we define

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

as the **line integral** or **contour integral** of $f$ along $C$ in terms of parameter $t$

**Properties** ✎

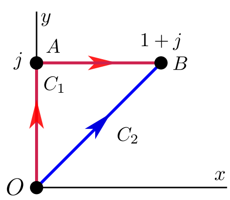
- $\int_C [z_0 f(z) + g(z)]dz = z_0 \int_C f(z)dz + \int_C g(z)dz, \quad z_0 \in \mathbf{C}$
- $\int_{-C} f(z)dz = -\int_C f(z)dz$
- $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$
- if $C$ is a simple closed path then we write $\int_C f(z)dz = \oint_C f(z)dz$






**example:** $f(z) = y - x - j3x^2 \quad (z = x + jy)$

- $I_1 = \int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$
  - segment $OA$: $z = 0 + jy, dz = jdy$
    $\int_{OA} f(z)dz = \int_0^1 (y - 0 - j0)jdy = j/2$
  - segment $AB$: $z = x + j, dz = dx$
    $\int_{AB} f(z)dz = \int_0^1 (1 - x - j3x^2)dx = 1/2 - j$




- $I_2 = \int_{C_2} f(z)dz$

$$z = x + jx, \quad dz = (1 + j)dx, \quad \int_{C_2} f(z)dz = \int_0^1 (x - x - j3x^2)(1 + j)dx = 1 - j$$

**remark:** $I_1 = \frac{1-j}{2} \neq I_2$ though $C_1$ and $C_2$ start and end at the same points

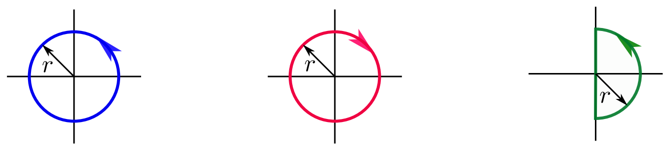


**example:** compute $\int_C \bar{z}dz$ on the following contours

the contour is a circle, so we write $z$ in polar form, and note that $r$ is unchanged

$$z = re^{j\theta}, \quad dz = jre^{j\theta}d\theta, \quad \theta_1 \leq \theta \leq \theta_2$$

$$I = \int_{\theta_1}^{\theta_2} \overline{re^{j\theta}} \cdot jre^{j\theta}d\theta = jr^2 \int_{\theta_1}^{\theta_2} 1\, d\theta$$

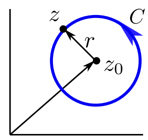



$$I = jr^2 \int_0^{2\pi} 1\, d\theta = j2\pi r^2$$

$$I = -jr^2 \int_0^{2\pi} 1\, d\theta = -j2\pi r^2$$

$$I = jr^2 \int_{-\pi/2}^{\pi/2} 1\, d\theta = j\pi r^2$$

**example:** let $C$ be a circle of radius $r$, centered at $z_0$




show that $\int_C (z-z_0)^m dz = \begin{cases} 0, & m \neq -1 \\ j2\pi, & m = -1 \end{cases}$

we parametrize the circle by writing

$$z = z_0 + re^{j\theta}, \quad 0 \leq \theta \leq 2\pi, \quad \text{so } dz = jre^{j\theta} d\theta$$

the integral becomes

$$I = \int_C r^m e^{jm\theta} \cdot jre^{j\theta} dz = jr^{m+1} \int_0^{2\pi} e^{j(m+1)\theta} d\theta$$

if $m = -1$, $I = j\int_0^{2\pi} d\theta = j2\pi$; otherwise, for $m \neq -1$, we have

$$I = jr^{m+1} \int_0^{2\pi} \{\cos[(m+1)\theta] + j\sin[(m+1)\theta]\} \; d\theta = 0$$



## Independence of path

under which condition does a contour integral only depend on the endpoints ?

**assumptions:**

- let $D$ be a domain and $f : D \to \mathbf{C}$ be a continuous function
- let $C$ be *any contour* in $D$ that starts from $z_1$ to $z_2$

we say $f$ has an **antiderivative** in $D$ if there exists $F : D \to \mathbf{C}$ such that

$$F'(z) = \frac{dF(z)}{dz} = f(z)$$

**Theorem:** if $f$ has an antiderivative $F$ on $D$, the contour integral is given by

$$\int_C f(z)dz = F(z_2) - F(z_1)$$



**example:** $f(z)$ is the principal branch

$$z^j = e^{j\,\mathrm{Log}\,z} \quad (|z| > 0, -\pi < \mathrm{Arg}\,z < \pi)$$

of this power function, compute the integral

$$\int_{-1}^{1} z^j dz$$

by two methods:

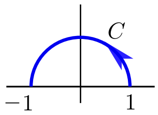
- using a parametrized curve $C$ which is the semicircle $z = e^{j\theta}$, $(0 \leq \theta \leq \pi)$
- using an antiderivative of $f$ of the branch

$$z^j = e^{j\log z} \quad (|z| > 0, -\pi/2 < \arg z < 3\pi/2)$$

**parametrized curve:** $z = e^{j\theta}$ and $dz = je^{j\theta}d\theta$

$$z^j = e^{j\log z} = e^{j(\mathrm{Log}\,1 + j\arg z)} = e^{j\cdot j\theta} = e^{-\theta}, \quad (0 < \theta < \pi)$$

the integral becomes




$$\begin{aligned} \int_C z^j dz &= \int_0^\pi j e^{(j-1)\theta} d\theta \\ &= \frac{j}{j-1} e^{(j-1)\theta}\Big|_0^\pi \\ &= \frac{j}{j-1}(e^{(j-1)\pi} - 1) = \frac{-j}{j-1}(e^{-\pi}+1) \\ &= -\frac{(1-j)(e^{-\pi}+1)}{2} \end{aligned}$$

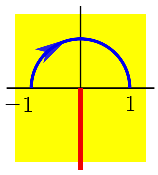
hence, $\int_{-1}^{1} z^j dz = \int_{-C} z^j dz = \dfrac{(1-j)(e^{-\pi 1}+1)}{2}$



**antiderivative of $z^j$ is** $z^{j+1}/(j+1)$ on the branch

$$z^j = e^{j\log z} \quad (|z| > 0, -\pi/2 < \arg z < 3\pi/2)$$

(we cannot use the principal branch because it is not defined at $z = -1$)



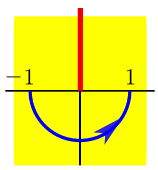

$$\begin{aligned} \int_{-1}^{1} z^j dz &= \left[\frac{z^{j+1}}{j+1}\right]_{-1}^{1} = \frac{1}{j+1}\left[1^{j+1} - (-1)^{j+1}\right] \\ &= \frac{1}{j+1}\left[e^{(j+1)\log 1} - e^{(j+1)\log(-1)}\right] \\ &= \frac{1}{j+1}\left[e^{(j+1)(\mathrm{Log}\,1 + j0)} - e^{(j+1)(\mathrm{Log}\,1 + j\pi)}\right] \\ &= \frac{1}{j+1}\left[1 - e^{j\pi - \pi}\right] = \frac{(1-j)(e^{-\pi}+1)}{2} \end{aligned}$$

the integral computed by the two methods are equal



if we use an antiderivative of $z^j$ on a different branch

$$z^j = e^{j\log z} \quad (|z| > 0, \pi/2 < \arg z < 5\pi/2)$$



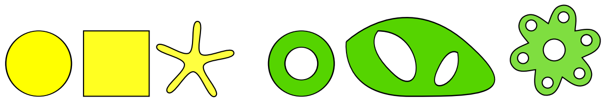

$$\begin{aligned} \int_{-1}^{1} z^j dz &= \frac{1}{j+1}\left[e^{(j+1)\log 1} - e^{(j+1)\log(-1)}\right] \\ &= \frac{1}{j+1}\left[e^{(j+1)(\mathrm{Log}\,1 + j2\pi)} - e^{(j+1)(\mathrm{Log}\,1 + j\pi)}\right] \\ &= \frac{1}{j+1}\left[e^{-2\pi + j2\pi} - e^{-\pi + j\pi}\right] \\ &= \frac{1}{j+1}\left[e^{-2\pi} + e^{-\pi}\right] \\ &= \frac{(1-j)e^{-\pi}(e^{-\pi}+1)}{2} \end{aligned}$$

the integral is different now as the function value of the integrand has changed

## Simply and Multiply connected domains

a **simply connected** domain $D$ is a domain such that every simple closed contour within it encloses only points of $D$

intuition: a domain is simply connected if it has no **holes**



simply connected multiply connected

a domain that is not simply connected is called **multiply connected**



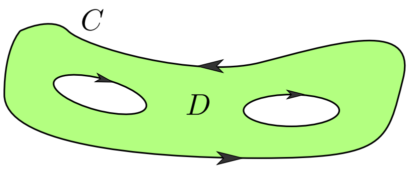
## Green's Theorem

let $D$ be a bounded domain whose boundary $C$ is sectionally smooth

let $P(x,y)$ and $Q(x,y)$ be *continuously differentiable* on $D \cup C$, then

$$\int_C P dx + \int_C Q dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where $C$ is in the positive direction w.r.t. the interior of $D$




$D$ can be simply or multiply connected

this result will be used to prove the Cauchy's theorem

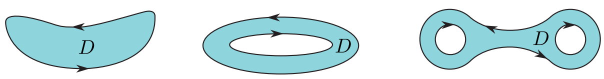


## Cauchy's theorem

let $D$ be a bounded domain whose boundary $C$ is sectionally smooth

**Theorem:** if $f(z)$ is analytic and $f'(z)$ is continuous **in** $D$ **and on** $C$ then

$$\int_C f(z) dz = 0$$




Goursat proved this result w/o the assumption on continuity of $f'$

the consequence is then known as the **Cauchy-Goursat theorem**

**Proof of Cauchy's theorem:**

$$f(z) = u(x,y) + jv(x,y), \quad dz = dx + jdy$$
$$f(z)dz = (u + jv)(dx + jdy) = u\ dx - v\ dy + j(v\ dx + u\ dy)$$

if $f'$ is continuous in $D$, so are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, then from Green's theorem

$$\int_C f(z)dz = \int\int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + j\int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

since $f$ is analytic, the Cauchy-Riemann equations suggest that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so we can conclude that

$$\int_C f(z)dz = 0$$



**example:** for *any* simple closed contour $C$

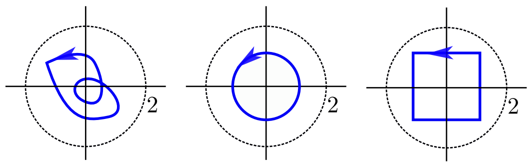
$$\int_C e^{z^2} dz = 0$$

because $e^{z^2}$ is a composite of $e^z$ and $z^2$, so $f$ is analytic everywhere

**example:** the integral

$$\int_C \frac{ze^z}{(z^2+4)^2} dz = 0$$

for any closed contour lying in the open disk $|z| < 2$






## Extension to multiply connected domains

let $D$ be a *multiply connected domain*

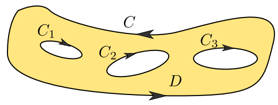
**Cauchy-Goursat theorem:** suppose that

1. $C$ is a simple closed contour in $D$, described in **counterclockwise** direction
2. $C_1, \ldots, C_n$ are simple closed contours interior to $C$, all in **clockwise** direction
3. $C_1, \ldots, C_n$ are **disjoint** and their interiors have no points in common

(then $D$ consists of the points in $C$ and exterior to each $C_k$)

if $f$ is analytic **on** *all of these contours* and **throughout** $D$ then

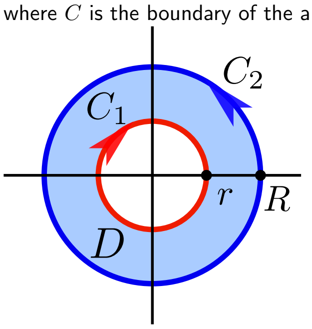
$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$



**example:** use the result from page 11-16 to compute

$$\int_C \frac{1}{z}\, dz$$

where $C$ is the boundary of the annulus $D$ shown below (where $r, R > 0$)




if $z = r^{j\theta}$ then $z\bar{z} = |z|^2 = r^2$

from p. 11-16, we obtain

$$\int_{C_2} R^2/z dz = j2\pi R^2, \quad \text{or that}$$

$$\int_{C_1} \frac{1}{z} dz = -j2\pi, \quad \int_{C_2} \frac{1}{z} dz = j2\pi$$

therefore, $\int_C \frac{1}{z}\, dz = \int_{C_1} \frac{1}{z}\, dz + \int_{C_2} \frac{1}{z} dz = 0$

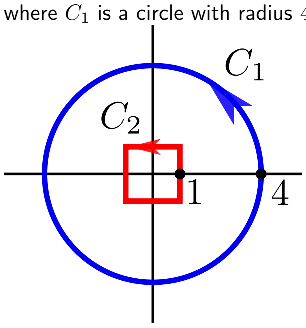
agree with the Cauchy's theorem since $f$ is analytic everywhere in $D$ and on $C$



**example:** for each $f$, use the Cauchy-Goursat theorem on p. 11-27 to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

where $C_1$ is a circle with radius $4$ and $C_2$ is a square shown below




$$f(z) = \frac{1}{3z^2+1}$$

$$f(z) = \frac{z+2}{\sin(z/2)}$$

$$f(z) = \frac{z}{1-e^z}$$

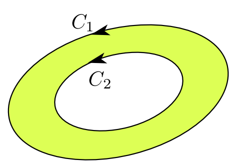
where are the singular points of these $f$?

this result is known as **the principle of deformation path**



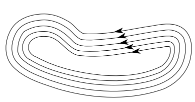
## Principle of deformation of paths

let $C_1$ and $C_2$ be **positively oriented** simple closed contours where $C_2$ is interior to $C_1$




**Theorem:** if a function $f$ is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



meaning: integrals of an analytic function does **not depend** on the path if the function is analytic in *between* and *on* the two paths

## Cauchy integral formula

let $C$ be a simple closed contour, taken in the positive sense

**Theorem:** let $f$ be analytic everywhere *inside* and *on* $C$

if $z_0$ is any point interior to $C$ then

$$f(z_0) = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)} dz$$

this is known as the **Cauchy integral formula**

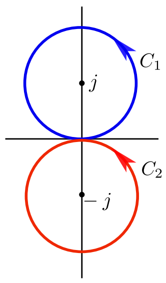
meaning: certain integrals along contours can be determined by the values of $f$



**example:** compute $\int_C \frac{e^z}{z^2+1} dz$ on the contours $C_1$ and $C_2$

$$\text{write} \quad \int_C \frac{e^z}{z^2+1} dz = \int_C \frac{e^z}{(z+j)(z-j)} dz$$

choose $f(z)$ such that it is analytic everywhere on each contour




- to compute $\int_{C_1} \frac{e^z}{z^2+1} dz$ choose $f(z) = e^z/(z+j)$

$$\int_{C_1} \frac{e^z}{(z+j)(z-j)} dz = j2\pi f(j) = \pi e^j$$

- to compute $\int_{C_2} \frac{e^z}{z^2+1} dz$ choose $f(z) = e^z/(z-j)$

$$\int_{C_1} \frac{e^z}{(z+j)(z-j)} dz = j2\pi f(-j) = -\pi e^{-j}$$



## Upper bounds for contour integrals

assumptions:

- $C$ denotes a contour of length $L$
- $f$ is piecewise continuous on $C$

✎ **Theorem:** if there exists a constant $M > 0$ such that

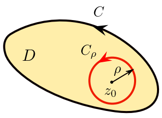
$$|f(z)| \leq M$$

for all $z$ on $C$ at which $f(z)$ is defined, then

$$\left| \int_a^b f(z) dz \right| \leq ML$$

## Proof of Cauchy integral formula

create a small circle $C_\rho$ which is interior to $C$




$f(z)$ is analytic everywhere in $D$

$\dfrac{f(z)}{z-z_0}$ is analytic in $D$ except at $z = z_0$

from the Cauchy-Goursat theorem,

$$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\, dz$$

which can be expressed as

$$\int_C \frac{f(z)}{z-z_0} dz - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz$$

we can show that $\int_{C_\rho} \frac{dz}{z-z_0} = j2\pi$ (similar to example on page 11-16)



therefore, we obtain

$$\int_C \frac{f(z)}{z-z_0} dz - j2\pi f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz$$

and we will show that the RHS must be zero

since $f$ is analytic, it is continuous at $z_0$, $e.g.$, for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

if we pick $\rho$ to be smaller than $\delta$ then $|f(z) - f(z_0)|/|z - z_0| < \varepsilon/\rho$

we can show that the integral is bounded by (from page 11-33)

$$\left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz \right| < \frac{\varepsilon \cdot \text{length of } C_\rho}{\rho} = 2\pi\varepsilon$$

since we can let $\varepsilon$ be arbitrarily small, the integral must be equal to zero



## Derivatives of analytic functions

let $D$ be a simply connected domain and $z_0$ be any interior point of $D$

**Theorem:** if $f$ is analytic in $D$ then the derivative of $f(z_0)$ of all order exist and are analytic in $D$

moreover, the derivatives of $f$ at $z$ are given by

$$f^{(n)}(z) = \frac{n!}{j2\pi}\int_C \frac{f(s)}{(s-z)^{n+1}}\, ds \qquad (n = 1, 2, \ldots)$$

**example:** compute $\int_C \frac{e^{2z}}{z^4} dz$ where $C$ is the positively oriented unit circle

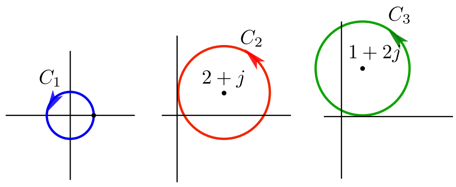
$$\int_C \frac{e^{2z}}{z^4}\, dz \triangleq \int_C \frac{f(z)}{(z-0)^{3+1}} dz = \frac{j2\pi f^{(3)}(0)}{3!} = \frac{j8\pi}{3}$$

(where $f(z) = e^{2z}$)

**example:** compute $\int_C f(z)dz$ where $f(z) = \dfrac{(z+1)}{(z^3 - 2z^2)}$

$C$ are circles given by $|z| = 1, \quad |z-2-j| = 2, \quad |z-1-j2| = 2$

(all are in counterclockwise direction)

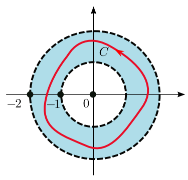



$$f_1(z) = \frac{z+1}{z-2}, \qquad f_2(z) = \frac{z+1}{z^2}, \qquad f_3(z) = \frac{z+1}{z^2(z-2)} = f(z)$$

$$\int_{C_1} f(z)\,dz = j2\pi f_1'(0), \; \int_{C_2} f(z)\,dz = j2\pi f_2(2), \; \int_{C_3} f(z)\,dz = 0$$



**example:** let $C$ be a simple closed contour lying in the annulus $1 < |z| < 2$




compute $\displaystyle \int_C \frac{3z^3 + 2z^2 - 8z - 4}{z^2(z^2+3z+2)} dz$

$f$ is not analytic at $0, -1, -2$, so the Cauchy formula cannot be readily applied

we can compute the partial fraction of $f$ and the integral becomes

$$\int_C f(z)dz = -\int_C \frac{1}{z} dz - \int_C \frac{1}{z^2} dz + \int_C \frac{3}{z+1} dz + \int_C \frac{1}{z+2} dz$$

applying the Cauchy integral formula to each term gives

$$\int_C f(z)dz = j2\pi(-1) + j2\pi(0) + j2\pi(3) + 0 = j4\pi$$



## References


Chapter 4 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 3 in

T. W. Gamelin, *Complex Analysis*, Springer, 2001

Chapter 22 in

M. Dejnakarin, *Mathematics for Electrical Engineering*, CU Press, 2006

## Exercises

1. $f(z) = \pi e^{\pi \bar{z}}$ and $C$ is the boundary of the square with vertices at the points $0, 1, 1+j$ and $j$, the orientation of $C$ being in the counterclockwise direction.

2. Find the following integrals where the path is any contour between the indicated limits of integration:

   (a) $\displaystyle\int_{j3}^{j/4} e^{\pi z}\, dz$

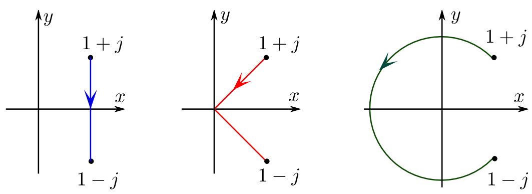
   (b) $\displaystyle\int_{0}^{\pi+j2} \sin z\, dz$

   (c) $\displaystyle\int_{2}^{4} (z-1)^2\, dz$

3. Directly evaluate the integral

$$\int_{1+j}^{1-j} (3z^2 + j2z)\, dz$$

   along the three paths joining the points $1+j$ and $1-j$ shown in the figure. Are the three values of the three the same ? Explain your reasons.




4. Evaluate the following integrals:

   (a) $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz,$

   (b) $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4}\, dz$

   where $C$ is the circle $|z| = 4$.

5. Evaluate the integral

$$\int_C \frac{dz}{z^2(z^2-4)e^z}$$

   where

   (a) $C$ is the circle $|z| = 1$,

   (b) $C$ is the circle $|z-1| = 2$.

6. Prove that if $f$ is analytic inside and on a circle $C$ of radius $r$ and center at $z = a$, then

$$|f^{(n)}(a)| \le \frac{M \cdot n!}{r^n}, \qquad n = 0, 1, 2, \ldots \tag{11.1}$$

   where $M$ is a constant such that $|f(z)| < M$. This is known as the Cauchy's inequality. Hint. Use the Cauchy's integral formula and apply an upperbound for the integral.

7. Liouville's theorem states that if for all $z$ in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e., $\exists M,\ |f(z)| < M$, then $f(z)$ must be a constant. Use the Cauchy's inequality from (11.1) to prove Liouville's theorem. Hint. Consider an upper bound for $|f'(z)|$ when $r \to \infty$.

8. Fundamental theorem of algebra states that any polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

of degree $n$ ($n > 1$) has at least one zero. That is, there exists at least one point $z_0$ such that $p(z_0) = 0$. Prove this theorem by using the result from Liouville's theorem. Hint. Consider $f(z) = 1/p(z)$. What happen to $f$ if $p(z) = 0$ has no root at all ?

9. By using the fundamental theorem of algebra, prove that every polynomial equation:

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n = 0,$$

where $n \geq 1$ and $a_n \neq 0$ has exactly $n$ roots.

# บทที่ 12

# Series

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถวิเคราะห์ได้ว่าฟังก์ชันเชิงซ้อนหนึ่งๆ นั้นมีอนุกรมเทย์เลอร์ (Taylor series) รอบจุดหนึ่งๆ หรือไม่ หากมี จะหาอนุกรมดังกล่าวได้อย่างไร
- สามารถหาอนุกรมโลรองต์ของฟังก์ชันเชิงซ้อน และอธิบายได้ว่าอนุกรมดังกล่าวมีผลใช้ได้ บนโดเมนใดในระนาบเชิงซ้อน

# 12. Series

- limit and convergence
- Taylor series
- Maclaurin series
- Laurent series



## Convergence of sequences

an infinite **sequence**

$$z_1, z_2, \dots, z_n, \dots$$

of complex numbers has a **limit** $z$, denoted by

$$\lim_{n\to\infty} z_n = z$$

if for each $\epsilon > 0$, there exists a positive integer $N$ such that

$$|z_n - z| < \epsilon \quad \text{whenver} \quad n > N$$

($z_n$ becomes arbitrarily close to $z$ as $n$ increases)

- if a limit exists it must be **unique**
- when the limit exists, the sequence is said to **converge to** $z$
- if the sequence has no limit, it **diverges**



## Limit of complex-valued sequences

suppose that $z_n = x_n + jy_n$ and $z = x + jy$; then

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

**example:** $z_n = \frac{1}{n^3} + j$ for $n = 1, 2, \dots$

$$\lim_{n\to} z_n = \lim_{n\to\infty} \frac{1}{n^3} + j \lim_{n\to\infty} 1 = 0 + j = j$$

moreover, we can see that for each $\epsilon > 0$

$$|z_n - j| = \frac{1}{n^3} \quad \text{whenver} \quad n > \frac{1}{\epsilon^{1/3}}$$

## Convergence of series

an infinite **series**

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_k + \cdots$$

of complex numbers **converges** to the **sum** $S$ if the sequence

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \ldots)$$

of **partial sums** converges to $S$; then we can write

$$\sum_{k=1}^{\infty} z_k = S \quad \text{if} \quad \lim_{n\to\infty} S_n = S$$

- a series can have **at most** one sum
- when a series does not converge, we say it **diverges**



## Limit of complex-valued series

suppose that $z_n = x_n + jy_n$ and $S = X + jY$; then

$$\sum_{n=1}^{\infty} z_k = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

**Facts:**

- if a series converges, the $n$th term converges to zero as $n \to \infty$
- the absolute convergence of a series implies the convergence of that series

$$\sum_{n=1}^{\infty} |z_n| \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} z_n \text{ converges}$$



**example:** the geometric series $\sum_{k=0}^{\infty} z^k$

the $n$th partial sum of the geometric series is given by

$$S_n = \sum_{k=0}^{n} z^k = 1 + z + z^2 + \cdots + z^{n-1} + z^n$$

multiply both sides by $1 - z$

$$(1-z)S_n = 1 - z + z - z^2 + \cdots + z^{n-1} - z^n + z^n - z^{n+1} = 1 - z^{n+1}$$

if $|z| < 1$ then $z^{n+1} \to 0$ and $S_n \to \dfrac{1}{1-z}$ as $n \to \infty$

the limit of the partial sum exists, and hence

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad |z| < 1$$

## Taylor series

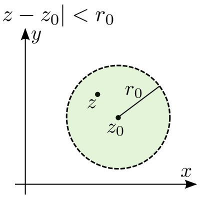
**Taylor's theorem:** suppose $f$ is **analytic** throughout a disk $|z - z_0| < r_0$ then $f(z)$ has the *power series* representation

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)(z - z_0)^2}{2!} + \cdots + \frac{f^{(n)}(z_0)(z - z_0)^n}{n!} + \cdots$$

for each $z$ inside the disk, $i.e.$, $|z - z_0| < r_0$

meaning: the power series converges to $f(z)$ when $|z - z_0| < r_0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}$$




the expansion of $f(z)$ is called the **Taylor series** of $f$ about the point $z_0$



## Maclaurin series

when $z_0 = 0$, the Taylor series becomes

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)z^n}{n!} \qquad (|z| < r_0)$$

and it is called a **Maclaurin series**

**example:** $f(z) = e^z$

since $e^z$ is entire, it has a Maclaurin representation that is valid for all $z$

$$f^{(n)} = e^z, \quad n = 0, 1, 2, \ldots, \quad \Longrightarrow \quad f^{(n)}(0) = 1 \text{ for all } n$$

and it follows that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$



**example:** Maclaurin representation of $f(z) = 1/(1 - z)$

$f(z)$ is analytic throughout the open disk $|z| < 1$ and its derivatives are

$$f^{(n)}(z) = \frac{n!}{(1 - z)^{n+1}} \quad \Longrightarrow \quad f^{(n)}(0) = n! \quad (n = 0, 1, 2, \ldots)$$

therefore, the Maclaurin series is given by

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

it is simply a **geometric series** where $z$ is the common ratio of adjacent terms

agree with the result on page 12-6

**example:** Maclaurin representation of $f(z) = \sin z$

we write $\sin z = \dfrac{e^{jz} - e^{-jz}}{j2}$ and note that $\sin z$ is entire

then we can use the Maclaurin series of $e^z$ for expanding $e^{\pm jz}$

$$\sin z = \frac{1}{j2}\left(\sum_{n=0}^{\infty}\frac{(jz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-jz)^n}{n!}\right) = \frac{1}{j2}\sum_{n=0}^{\infty}[1-(-1)^n]\frac{j^n z^n}{n!}$$

but $(1-(-1)^n) = 0$ when $n$ is even and $2$ otherwise, so we replace $n$ by $2n+1$

$$\sin z = \frac{1}{j2}\sum_{n=0}^{\infty}\frac{2j^{2n+1}z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$
$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

the series contains only **odd** powers of $z$



**Maclaurin series expansion**

for $|z| < \infty$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{(2n+1)}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{(2n)}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty}\frac{z^{(2n+1)}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

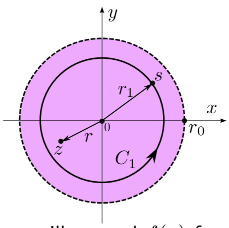
$$\cosh z = \sum_{n=0}^{\infty}\frac{z^{(2n)}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$



## Proof of Taylor's theorem

assumption: $f$ is analytic on $|z| < r_0$

proof for special case: $z_0 = 0$; $\quad f(z) = \displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)z^n}{n!} \quad (|z| < r_0)$




- $C_1$ is a positively oriented circle $|z| = r_1$
- $z$ is any point with $|z| = r$ and $r < r_1 < r_0$
- $s$ is a point on contour $C_1$
- $f$ is analytic *inside* and *on* the circle $C_1$

we will expand $f(z)$ from the Cauchy integral formula

$$f(z) = \frac{1}{j2\pi}\int_{C_1}\frac{f(s)}{s-z}dz$$

**expand the integral term**

- rewrite $1/(s-z)$ as $\dfrac{1}{s-z} = \dfrac{1}{s} \cdot \dfrac{1}{1-(z/s)}$
- for any $z \neq 1$,

$$\frac{1}{1-z} = \frac{z^N}{1-z} + \sum_{n=0}^{N-1} z^n \qquad \text{(from long division)}$$

- then we can write

$$\frac{1}{s-z} = \frac{z^N}{s^N(s-z)} + \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}}$$

- multiply by $f(s)$ and integrate with respect to $s$ along $C_1$

$$\int_{C_1} \frac{f(s)}{s-z}\, ds = z^N \int_{C_1} \frac{f(s)}{(s-z)s^N}\, ds + \sum_{n=0}^{N-1} z^n \int_{C_1} \frac{f(s)}{s^{n+1}}\, ds$$

Series 12-13

**characterize the remainder term**

- the second term on RHS can be computed from the Cauchy integral formula

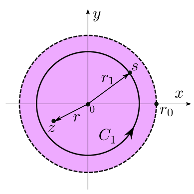
$$\int_{C_1} \frac{f(s)}{s^{n+1}} ds = j2\pi \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots)$$

- from $f(z) = \frac{1}{j2\pi} \int_{C_1} \frac{f(s)}{s-z} ds$, we obtain

$$f(z) = \underbrace{\frac{z^N}{j2\pi} \int_{C_1} \frac{f(s)}{s^N(s-z)}\, ds}_{R_N(z)} + \sum_{n=0}^{N-1} \frac{f^{(n)}(0) z^n}{n!}$$

- we obtain Taylor's representation if we can show that $\lim_{N\to\infty} R_N(z) = 0$

Series 12-14




**the remainder term goes to zero as $n \to \infty$**

- $|s-z| \geq ||s| - |z|| = r_1 - r$; hence $1/(s-z) \leq 1/(r_1 - r)$
- if $|f(s)| \leq M$ on $C_1$ then

$$|R_N(z)| = \left|\frac{z^N}{j2\pi}\right| \left| \int_{C_1} \frac{f(s)}{s^N(s-z)} ds \right| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_1 - r) r_1^N} \cdot \underbrace{\text{length of } C_1}_{2\pi r_1}$$

$$= \left(\frac{r}{r_1}\right)^N \frac{M r_1}{(r_1 - r)} \to 0, \quad \text{as } N \to \infty \text{ because } r/r_1 < 1$$

- we finished the proof for the special case of Taylor's theorem; when $z_0 = 0$

Series 12-15

**generalize the result to** $z_0 \neq 0$

assumption: $f$ is analytic on $|z - z_0| < r_0$

- $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < r_0$ (composite function)
- hence, $g(z) = f(z + z_0)$ is analytic on $|z| < r_0$, so its Maclaurin series is

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!} \qquad (|z| < r_0)$$

- this is equivalent to

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0) z^n}{n!} \qquad (|z| < r_0)$$

- replace $z$ by $z - z_0$, we obtain the Taylor's series



**example:** expand $f(z) = \dfrac{1 + 2z}{z^3 + z^2}$ to a series involving powers of $z$

we cannot find a Maclaurin series for $f$ since it is not analytic at $z = 0$

however, for $|z| \neq 0$, we can write

$$\begin{aligned}
f(z) &= \frac{1}{z^2} \cdot \frac{1 + 2z}{1 + z} = \frac{1}{z^2} \cdot \left(2 - \frac{1}{1 + z}\right) \\
&= \frac{1}{z^2} \cdot \left(2 - (1 - z + z^2 - z^3 + z^4 - \cdots)\right) \qquad (|z| < 1) \\
&= \frac{1}{z^2}(1 + z - z^2 + z^3 - z^4 + \cdots) \qquad (0 < |z| < 1) \\
&= \frac{1}{z^2} - \frac{1}{z} + z - z^2 + \cdots
\end{aligned}$$

the expansion of $f$ contains both *negative* and *positive* powers of $z$



**remarks:**

- if $f$ fails to be analytic at a point $z_0$, we cannot apply Taylor's theorem there
- example in page 12-17 shows that however, it is possible to find a series for $f(z)$ involving both *positive* and *negative* powers of $(z - z_0)$

$$f(z) = \frac{1 + 2z}{z^3 + z^2} = \frac{1}{z^2} - \frac{1}{z} + z - z^2 + \cdots$$

- such a representation is known as **Laurent series**, which includes the Taylor series as a special case
- with the Laurent series, we can expand $f$ about a singular point

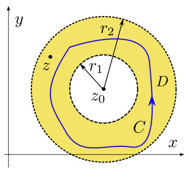
## Laurent series

**Theorem:** if all of the following assumptions hold

1. $D$ is an **annular** domain $r_1 < |z - z_0| < r_2$
2. $C$ is any positively oriented simple closed contour around $z_0$ and lies inside $D$
3. $f$ is analytic throughout $D$

then $f$ has the series representation; called the **Laurent series**

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\ \text{where} \quad a_n &= \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad (n = 0, 1, \ldots) \\ b_n &= \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz, \quad (n = 1, 2, \ldots) \end{aligned}$$






**remarks:**

- we cannot apply the Cauchy integral formula to compute the coefficient $a_n$

$$a_n = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

  because $f$ is NOT analytic in $C$
- if the annular domain is specified, a Laurent series of $f(z)$ about $z_0$ is unique
- the annulus $D$ is the region of convergence for the obtained Laurent series
- the coeff $a_n$ and $b_n$ given by the formula are generally difficult to compute
- so, we use another way such as computing a partial fraction of $f$ and use

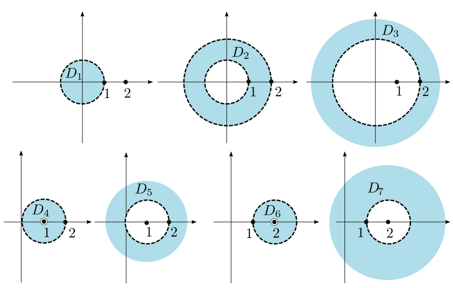
$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

  to expand the partial fraction as an infinite series



**example:** find power series representation of $f(z) = \dfrac{-1}{(z - 1)(z - 2)}$ in

$$D_1 : |z| < 1, \qquad D_2 : 1 < |z| < 2, \qquad D_3 : 2 < |z| < \infty$$
$$D_4 : 0 < |z - 1| < 1, \quad D_5 : 1 < |z - 1|, \quad D_6 : 0 < |z - 2| < 1, \quad D_7 : 1 < |z - 2|$$




$f$ is not analytic at $z = 1$ and $z - 2$

- **domain** $D_1$**:** $|z| < 1$ ($|z| < 1$ and $|z/2| < 1$ for all $z \in D_1$ )

$$f(z) = f(z) = \frac{-1}{1-z} + \frac{1/2}{1-(z/2)}$$
$$= -\sum_{n=0}^{\infty} z^n + (1/2)\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(2^{-n-1}-1)z^n, \qquad |z| < 1$$

the representation is a Maclaurin series

- **domain** $D_2$**:** $1 < |z| < 2$ ($|1/z| < 1$ and $|z/2| < 1$ for all $z \in D_2$)

$$f(z) = \frac{1}{z}\cdot\frac{1}{1-(1/z)} + \frac{1}{2}\cdot\frac{1}{1-(z/2)}$$
$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \qquad (1 < |z| < 2)$$

this is *the* Laurent series for $f$ in $D_2$ where $a_n = 1/2^{n+1}$ and $b_n = 1$



- **domain** $D_3$**:** $2 < |z| < \infty$ ( $|2/z| < 1$ and so $|1/z| < 1$ for all $z \in D_3$)

$$f(z) = \frac{1}{z}\cdot\frac{1}{1-(1/z)} - \frac{1}{z}\cdot\frac{1}{1-(2/z)}$$
$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{(1-2^{n-1})}{z^n}, \qquad (2 < |z| < \infty)$$

this is *the* Laurent series for $f$ in $D_3$ where $a_n = 0$ and $b_n = 1 - 2^{n-1}$

- **domain** $D_4$**:** $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-1} + \frac{1}{1-(z-1)}$$
$$= \frac{1}{z-1} + \sum_{n=0}^{\infty}(z-1)^n \quad (0 < |z-1| < 1)$$

this is *the* Laurent series for $f$ in $D_4$ where $b_1 = 1, b_k = 0, k \geq 2$ and $a_n = 1$



- **domain** $D_5$**:** $1 < |z-1|$ ( $1/|z-1| < 1$ for all $z \in D_5$)

$$f(z) = \frac{-1}{(z-1)(z-1-1)} = \frac{-1}{(z-1)^2}\cdot\frac{1}{1-1/(z-1)}$$
$$= -\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+2}}, \qquad (1 < |z-1| < \infty)$$

this is *the* Laurent series for $f$ in $D_5$ where $a_n = 0$, $b_1 = 0, b_n = -1, n \geq 2$

- **domain** $D_6$**:** $0 < |z-2| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{(1+z-2)} - \frac{1}{z-2}$$
$$= -\frac{1}{z-2} + \sum_{n=0}^{\infty}(-1)^n(z-2)^n \qquad (0 < |z-2| < 1)$$

this is *the* Laurent series for $f$ in $D_4$ with $b_1 = -1, b_n = 0, n \geq 2$, $a_n = (-1)^n$

- **domain** $D_7$**:** $1 < |z-2|$ ( $1/|z-2| < 1$ for all $z \in D_7$)

$$f(z) = \frac{-1}{(z-2+1)(z-2)} = \frac{-1}{(z-2)^2} \cdot \frac{1}{1+1/(z-2)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(z+2)^{n+2}}, \qquad (1 < |z-2| < \infty)$$

*the* Laurent series for $f$ in $D_7$ where $a_n = 0,\ b_1 = 0, b_n = (-1)^{n+1}, n \geq 2$

**remark:** we can find related integrals from the coefficients of the Laurent series

for example, let $C$ be a simple positive closed contour lying in $D_7$

$$\begin{array}{lllllll} \int_C \frac{-1}{(z-1)(z-2)}\,dz & = & \int_C f(z)dz & = & j2\pi b_1 & = & 0 \\ \int_C \frac{-1}{(z-1)(z-2)^2}\,dz & = & \int_C \frac{f(z)}{(z-2)}dz & = & j2\pi a_0 & = & 0 \\ \int_C \frac{-1}{(z-1)}\,dz & = & \int_C f(z)(z-2)dz & = & j2\pi b_2 & = & -j2\pi \end{array}$$



**example:** find a Laurent series for $f(z) = \dfrac{e^z}{(z+1)^2}$ in a certain domain

for any $z$, since $e^z$ has a Maclaurin series about $0$, we can write

$$\begin{aligned} \frac{e^z}{(z+1)^2} &= \frac{e^{z+1}}{e(z+1)^2} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^n}{n!(z+1)^2} \\ &= \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^{n-2}}{n!} \\ &= \frac{1}{e}\left[\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right], \qquad (0 < |z+1| < \infty) \end{aligned}$$

this is the Laurent series for $f$ in the domain $0 < |z+1| < \infty$ where

$$b_1 = 1/e, \quad b_2 = 1/e, \quad b_k = 0, \forall k \geq 3, \quad a_n = \frac{1/e}{(n+2)!}$$



## References

Chapter 5 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 22 in

M. Dejnakarin, *Mathematics for Electrical Engineering*, CU Press, 2006

## Exercises

1. Find Maclaurin series of

   (a) $\dfrac{1}{(1-z)^2}$

   (b) $\dfrac{2}{(1+z)^3}$

   in certain domains. Specify those domains and express the series as $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Give a general expression of $a_n$ as a function of $n$.

2. Write the two Laurent series in powers of $z$ that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

   in certain domains, and specify those domains.

3. Show that

$$\frac{e^z}{1+z} = 1 + \frac{z^2}{2} - \frac{z^3}{3} + \frac{3z^4}{8} - \frac{11z^5}{30} + \cdots$$

   in a certain domain. Specify that domain. Show that the general term of the power series is given by

$$a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right], \quad n \geq 2.$$

4. Find all the possible Laurent series of

$$f(z) = \frac{1}{(z+1)(z+3)}.$$

   Specify the domains where the expansions are valid.

5. Find the Laurent series about $z = -2$ of

$$f(z) = (z-3)\sin\frac{1}{z+2}.$$

   Specify the domain where the expansion is valid.

# บทที่ 13

# Residue Theorem

วัตถุประสงค์การเรียนรู้ของเนื้อหาในบทนี้คือ

- สามารถหาค่าเรซิดิวของฟังก์ชันที่จุดเอกฐานหนึ่งๆ ได้ โดยการหาจากอนุกรมโลรองต์ หรือ จากสูตรเรซิดิว
- สามารถประยุกต์ทฤษฎีบทเรซิดิวโคชี (Cauchy's residue theorem) ในการหาอินทิกรัลบนเส้นรอบของปิดได้
- สามารถประยุกต์ทฤษฎีบทเรซิดิวโคชี ในการหาอินทิกรัลที่มีปริพัทธ์ที่เกี่ยวข้องกับฟังก์ชันไซน์หรือโคไซน์ และ อินทิกรัลไม่เหมาะสม (improper integral) และการแปลงกลับของลาปลาซ (inversion of Laplace transform) ได้

# 13. Residues and Its Applications

- isolated singular points
- residues
- Cauchy's residue theorem
- applications of residues



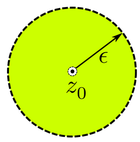
## Isolated singular points

$z_0$ is called a **singular point** of $f$ if

- $f$ fails to be analytic at $z_0$
- but $f$ is analytic at *some* point in *every* neighborhood of $z_0$

a singular point $z_0$ is said to be **isolated** if $f$ is analytic in *some* punctured disk

$$0 < |z - z_0| < \epsilon$$




centered at $z_0$ (also called a *deleted neighborhood* of $z_0$)

**example:** $f(z) = 1/(z^2(z^2+1))$ has the three isolated singular points at

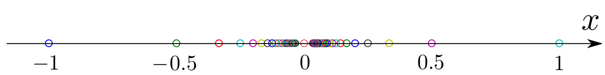
$$z = 0, \quad z = \pm j$$



## Non-isolated singular points

**example:** the function $\dfrac{1}{\sin(\pi/z)}$ has the singular points

$$z = 0, \quad z = \frac{1}{n}, \quad (n = \pm 1, \pm 2, \ldots)$$




- each singular point except $z = 0$ is isolated
- $0$ is nonisolated since *every* punctured disk of $0$ contains other singularities
- for any $\varepsilon > 0$, we can find a positive integer $n$ such that $n > 1/\varepsilon$
- this means $z = 1/n$ always lies in the punctured disk $0 < |z| < \varepsilon$

## Residues

assumption: $z_0$ is an isolated singular point of $f$, $e.g.$,

there exists a punctured disk $0 < |z - z_0| < r_0$ throughout which $f$ is analytic

consequently, $f$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots, \quad (0 < |z - z_0| < r_0)$$

let $C$ be any positively oriented simple closed contour lying in the disk

$$0 < |z - z_0| < r_0$$

the coefficient $b_n$ of the Laurent series is given by

$$b_n = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz, \qquad (n = 1, 2, \ldots)$$



the coefficient of $1/(z - z_0)$ in the Laurent expansion is obtained by

$$\int_C f(z)dz = j2\pi b_1$$

$b_1$ is called the **residue** of $f$ at the **isolated singular point** $z_0$, denoted by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

this allows us to write

$$\int_C f(z)dz = j2\pi \operatorname*{Res}_{z=z_0} f(z)$$

which provides a powerful method for evaluating integrals around a contour



**example:** find $\int_C e^{1/z^2} dz$ when $C$ is the positive oriented circle $|z| = 1$

$1/z^2$ is analytic everywhere except $z = 0$; $0$ is an isolated singular point

the Laurent series expansion of $f$ is

$$f(z) = e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \cdots \quad (0 < |z| < \infty)$$

the residue of $f$ at $z = 0$ is zero ($b_1 = 0$), so the integral is zero

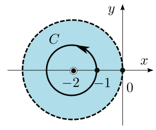
**remark:** the analyticity of $f$ within and on $C$ is a *sufficient condition* for $\int_C f(z)dz$ to be zero; however, it is not a *necessary condition*

**example:** compute $\int_C \frac{1}{z(z+2)^3}\,dz$ where $C$ is circle $|z+2| = 1$

$f$ has the isolated singular points at $0$ and $-2$

choose an annulus domain: $0 < |z+2| < 2$

on which $f$ is analytic and contains $C$




$f$ has a Laurent series on this domain and is given by

$$f(z) = \frac{1}{(z+2-2)(z+2)^3} = -\frac{1}{2}\cdot\frac{1}{1-(z+2)/2}\cdot\frac{1}{(z+2)^3}$$
$$= -\frac{1}{2(z+2)^3}\sum_{n=0}^{\infty}\frac{(z+2)^n}{2^n} = -\sum_{n=0}^{\infty}\frac{(z+2)^{n-3}}{2^{n+1}}, \quad (0 < |z+2| < 2)$$

the residue of $f$ at $z=-2$ is $-1/2^3$ which is obtained when $n=2$

therefore, the integral is $j2\pi(-1/2^3) = -j\pi/4$ (check with the Cauchy formula)



## Cauchy's residue theorem

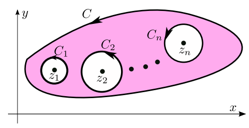
let $C$ be a positively oriented simple closed contour

**Theorem:** if $f$ is analytic inside and on $C$ except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside $C$, then

$$\int_C f(z)dz = j2\pi\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)$$

**Proof.**

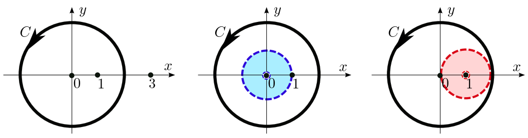
- since $z_k$'s are isolated points, we can find small circles $C_k$'s that are mutually disjoint
- $f$ is analytic on a multiply connected domain
- from the Cauchy-Goursat theorem:
  $\int_C f(z)dz = \sum_{k=1}^{n}\int_{C_k} f(z)dz$






**example:** use the Cauchy residue theorem to evaluate the integral

$$\int_C \frac{3(z+1)}{z(z-1)(z-3)}dz, \quad C \text{ is the circle } |z| = 2\text{, in counterclockwise}$$




$C$ encloses the two singular points of the integrand, so

$$I = \int_C f(z)dz = \int_C \frac{3(z+1)}{z(z-1)(z-3)}dz = j2\pi\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right]$$

- calculate $\operatorname{Res}_{z=0} f(z)$ via the Laurent series of $f$ in $0 < |z| < 1$
- calculate $\operatorname{Res}_{z=1} f(z)$ via the Laurent series of $f$ in $0 < |z-1| < 1$

rewrite $f(z) = \dfrac{1}{z} - \dfrac{3}{z-1} + \dfrac{2}{z-3}$

- the Laurent series of $f$ in $0 < |z| < 1$

$$f(z) = \frac{1}{z} + \frac{3}{1-z} - \frac{2}{3(1-z/3)} = \frac{1}{z} + 3(1+z+z^2+\dots) - \frac{2}{3}(1+(z/3)+(z/3)^2+\dots)$$

the residue of $f$ at $0$ is the coefficient of $1/z$, so $\mathrm{Res}_{z=0} f(z) = 1$

- the Laurent series of $f$ in $0 < |z-1| < 1$

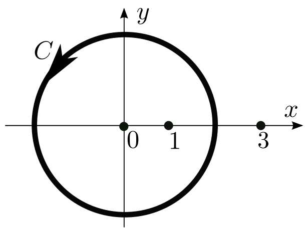
$$f(z) = \frac{1}{1+z-1} - \frac{3}{z-1} - \frac{1}{1-(z-1)/2}$$
$$= 1 - (z-1) + (z-1)^2 + \dots - \frac{3}{z-1} - \left(1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \dots\right)$$

the residue of $f$ at $1$ is the coefficient of $1/(z-1)$, so $\mathrm{Res}_{z=0} f(z) = -3$



therefore, $I = j2\pi(1-3) = -j4\pi$

alternatively, we can compute the integral from the Cauchy integral formula



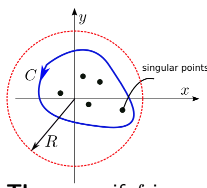

$$I = \int_C \left(\frac{1}{z} - \frac{3}{z-1} + \frac{2}{z-3}\right) dz$$
$$= j2\pi(1-3+0) = -j4\pi$$



## Residue at infinity

$f$ is said to have an **isolated point at** $z_0 = \infty$ if

there exists $R > 0$ such that $f$ is analytic for $R < |z| < \infty$



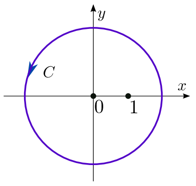

$C$ is a positive oriented simple closed contour

**Theorem:** if $f$ is analytic everywhere *except* for a finite number of singular points interior to $C$, then

$$\int_C f(z)dz = j2\pi \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

(see a proof on section 71, Churchill)

**example:** find $I = \int_C \frac{z-3}{z(z-1)} dz$, $C$ is the circle $|z| = 2$ (counterclockwise)




$$I = j2\pi \operatorname*{Res}_{z=0} \left[(1/z^2) f(1/z)\right]$$
$$= j2\pi \operatorname*{Res}_{z=0} \left[\frac{1-3z}{z(1-z)}\right] \triangleq j2\pi \operatorname*{Res}_{z=0} g(z)$$

find the residue via the Laurent series of $g$ in $0 < |z| < 1$

write $g(z) = \left(\frac{1}{z} - 3\right)(1 + z + z^2 + \cdots) \quad \Longrightarrow \quad \operatorname*{Res}_{z=0} g(z) = 1$

compare the integral with other methods ✎

- Cauchy integral formula (write the partial fraction of $f$)
- Cauchy residue theorem (have to find two residues; hence two Laurent series)



## Principal part

$f$ has an isolated singular point at $z_0$, so $f$ has a Laurent seires

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

in a punctured disk $0 < |z - z_0| < R$

the portion of the series that involves **negative powers** of $z - z_0$

$$\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

is called the **principal part of** $f$



## Types of isolated singular points

**three** possible types of the principal part of $f$

- no principal part

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \quad (0 < |z| < \infty)$$

- finite number of terms in the principal part

$$f(z) = \frac{1}{z^2(1+z)} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots, \quad (0 < |z| < 1)$$

- infinite number of terms in the principal part

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \quad (0 < |z| < \infty)$$

classify the number of terms in the principal part in a general form

- none: $z_0$ is called a **removable singular point**

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

- finite ($m$ terms): $z_0$ is called a **pole of order** $m$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

- infinite: $z_0$ is said to be an **essential singular point of** $f$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$



**examples:**

$$\begin{aligned} f_1(z) &= \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\ f_2(z) &= \frac{3}{(z-1)(z-2)} = -\left(\frac{1}{z-2} + 1 + (z-2) + (z-2)^3 + \cdots\right) \\ f_3(z) &= \frac{1}{z^2(1+z)} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots \\ f_4(z) &= e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \end{aligned}$$

- $0$ is a removeable singular point of $f_1$
- $2$ is a pole of order $1$ (or **simple pole**) of $f_2$
- $0$ is a pole of order $2$ (or **double pole**) of $f_3$
- $0$ is an essential singular point of $f_4$

**note:** for $f_2, f_3$ we can determine the pole/order from the denominator of $f$



## Residue formula

if $f$ has a pole of order $m$ at $z_0$ then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

**Proof.** if $f$ has a pole of order $m$, its Laurent series can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{m+n} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \cdots + b_m$$

to obtain $b_1$, we take the $(m-1)$th derivative and take the limit $z \to z_0$

**example 1:** find $\text{Res}_{z=0} f(z)$ and $\text{Res}_{z=2} f(z)$ where $f(z) = \dfrac{(z+1)}{z^2(z-2)}$

$$\begin{aligned}\operatorname*{Res}_{z=0} f(z) &= \lim_{z\to 0} \frac{d}{dz}\left(\frac{z+1}{z-2}\right) = -3/4 \qquad \text{(0 is a double pole of } f\text{)} \\ \operatorname*{Res}_{z=2} f(z) &= \lim_{z\to 2} \frac{z+1}{z^2} = 3/4\end{aligned}$$

**example 2:** find $\text{Res}_{z=0}\, g(z)$ where $g(z) = \dfrac{z+1}{1-2z}$

$g$ is analytic at $0$ ($0$ is a removable singular point of $g$), so $\text{Res}_{z=0}\, g(z) = 0$

**check** ✎ apply the results from the above two examples to compute

$$\int_C \frac{(z+1)}{z^2(z-2)} dz, \quad C \text{ is the circle } |z| = 3 \text{ (counterclockwise)}$$

by using the Cauchy residue theorem and the formula on page 13-12



sometimes the pole order cannot be readily determined

**example 3:** find $\text{Res}_{z=0} f(z)$ where $f(z) = \dfrac{\sinh z}{z^4}$

use the Maclaurin series of $\sinh z$

$$f(z) = \frac{1}{z^4}\cdot\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = \left(\frac{1}{z^3} + \frac{1}{3!z} + \frac{z}{5!} + \cdots\right)$$

$0$ is the **third**-order pole with residue $1/3!$

here we determine the residue at $z=0$ from its definition (the coeff. of $1/z$ )

no need to use the residue formula on page 13-18



when the pole order ($m$) is unknown, we can

- assume $m = 1, 2, 3, \ldots$
- find the corresponding residues until we find the first finite value

**example 4:** find $\text{Res}_{z=0} f(z)$ where $f(z) = \dfrac{1+z}{1-\cos z}$

- assume $m = 1$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{z(1+z)}{1-\cos z} = 0/0 = \lim_{z\to 0} \frac{1+2z}{\sin z} = 1/0 = \infty \implies \text{(not 1st order)}$$

- assume $m = 2$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2(1+z)}{1-\cos z}\right) = 2 \text{ (finite)} \quad \implies \quad 0 \text{ is a double pole}$$

**note:** use L'Hôpital's rule to compute the limit

## Summary

many ways to compute a contour integral ($\int_C f(z)dz$)

- parametrize the path (feasible when $C$ is easily described)
- use the principle of deformation of paths (if $f$ is analytic in the region between the two contours)
- use the Cauchy integral formula (typically requires the partial fraction of $f$)
- use the Cauchy's residue theorem on page 13-8 (requires the residues at singular points enclosed by $C$)
- use the theorem of the residue at infinity on page 13-12 (find one residue at $0$)

to find the residue of $f$ at $z_0$

- read from the coeff of $1/(z - z_0)$ in the Laurent series of $f$
- apply the residue formula on page 13-18



## Application of the residue theorem

- calculating real definite integrals
  - integrals involving sines and cosines
  - improper integrals
  - improper integrals from Fourier series
- inversion of Laplace transforms



## Definite integrals involving sines and cosines

we consider a problem of evaluating definite integrals of the form

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$$

since $\theta$ varies from $0$ to $2\pi$, we can let $\theta$ be an argument of a point $z$

$$z = e^{j\theta} \quad (0 \leq \theta \leq 2\pi)$$

this describe a positively oriented circle $C$ centered at the origin

make the substitutions

$$\sin\theta = \frac{z - z^{-1}}{j2}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{jz}$$

this will transform the integral into the *contour* integral

$$\int_C F\left(\frac{z-z^{-1}}{j2}, \frac{z+z^{-1}}{2}\right)\frac{dz}{jz}$$

- the integrand becomes a function of $z$
- if the integrand reduces to a rational function of $z$, we can apply the Cauchy's residue theorem

**example:**

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} &= \int_C \frac{1}{5+4\frac{(z-z^{-1})}{2j}}\frac{dz}{jz} = \int_C \frac{dz}{2z^2+j5z-2} \triangleq \int_C g(z)dz \\ &= \int_C \frac{dz}{2(z+2j)(z+j/2)} = j2\pi\left(\operatorname*{Res}_{z=-j/2} g(z)\right) = 2\pi/3\end{aligned}$$

where $C$ is the positively oriented circle $|z|=1$



the above idea can be summarized in the following theorem

**Theorem:** if $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ which is finite on the closed interval $0 \leq \theta \leq 2\pi$, and if $f$ is the function obtained from $F(\cdot,\cdot)$ by the substitutions

$$\cos\theta = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{j2}$$

then

$$\int_C^{2\pi} F(\cos\theta,\sin\theta)\, d\theta = j2\pi\left(\sum_k \operatorname*{Res}_{z=z_k}\frac{f(z)}{jz}\right)$$

where the summation takes over all $z_k$'s that lie within the circle $|z|=1$



**example:** compute $I = \displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta, \quad -1<a<1$

make change of variables

- $\cos 2\theta = \dfrac{e^{j2\theta}+e^{-j2\theta}}{2} = \dfrac{z^2+z^{-2}}{2} = \dfrac{z^4+1}{2z^2}$
- $1-2a\cos\theta+a^2 = 1-2a(z+z^{-1})/2+a^2 = -\dfrac{az^2-(a^2+1)z+a}{z}$

we have $\int_0^{2\pi}F(\theta)d\theta = \int_C \frac{f(z)}{jz}dz \triangleq \int_C g(z)dz$ where

$$g(z) = -\frac{(z^4+1)z}{jz\cdot 2z^2(az^2-(a^2+1)z+a)} = \frac{(z^4+1)}{j2z^2(1-az)(z-a)}$$

we see that only the poles $z=0$ and $z=a$ lie inside the unit circle $C$

therefore, the integral becomes

$$I = \int_C g(z)dz = j2\pi \left( \operatorname*{Res}_{z=0} g(z) + \operatorname*{Res}_{z=a} g(z) \right)$$

- note that $z = 0$ is a double pole of $g(z)$, so

$$\operatorname*{Res}_{z=0} g(z) = \lim_{z=0} \frac{d}{dz}(z^2 g(z)) = -\frac{1}{j2} \cdot \frac{a^2+1}{a^2}$$

- $\operatorname*{Res}_{z=a} g(z) = \lim_{z=a} (z-a)g(z) = \frac{1}{j2} \cdot \frac{a^4+1}{a^2(1-a^2)}$

hence, $I = \frac{2\pi a^2}{1-a^2}$



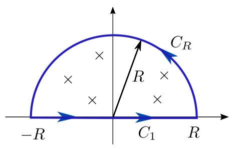
## Improper integrals

let's first consider a well-known improper integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

of course, this can be evaluated using the inverse tangent function

we will derive this kind of integral by means of **contour integration**




some poles of the integrand lie in the upper half plane

let $C_R$ be a semicircular contour with radius $R \to \infty$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = j2\pi \sum_k \operatorname*{Res}_{z=z_k} f(z)$$

and show that $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$

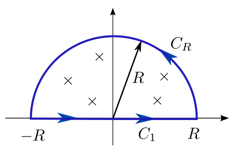


**Theorem:** if all of the following assumptions hold

1. $f(z)$ is analytic in the upper half plane except at a finite number of poles
2. none of the poles of $f(z)$ lies on the real axis
3. $|f(z)| \leq \frac{M}{R^k}$ when $z = Re^{j\theta}$; $M$ is a constant and $k > 1$

then the real improper integral can be evaluated by a contour integration, and

$$\int_{-\infty}^{\infty} f(x)dx = j2\pi \left[ \begin{array}{l} \text{sum of the residues of } f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$




- assumption 2: $f$ is analytic on $C_1$
- assumption 3: $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$

**Proof.** consider a semicircular contour with radius $R$ large enough to include all the poles of $f(z)$ that lie in the upper half plane

- from the Cauchy's residue theorem

$$\int_{C_1 \cup C_R} f(z)dz = j2\pi \left[\sum \operatorname{Res} f(z) \text{ at all poles within } C_1 \cup C_R\right]$$

(to apply this, $f(z)$ cannot have singular points on $C_1$, $i.e.$, the real axis)

- the integral along the real axis is our desired integral

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{C_R} f(z)dz = \lim_{R\to\infty} \int_{C_1\cup C_R} f(z)dz$$

- hence, it suffices to show that

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \quad \text{by using } |f(z)| \leq M/R^k, \text{ where } k > 1$$



- apply the modulus of the integral and use $|f(z)| \leq M/R^k$

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{M}{R^k} \cdot \text{length of } C_R = \frac{M\pi R}{R^k}$$

hence, $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$ if $k > 1$

**remark:** an example of $f(z)$ that satisfies all the conditions in page 13-30

$$f(x) = \frac{p(x)}{q(x)}, \quad p \text{ and } q \text{ are polynomials}$$

$q(x)$ has **no real roots** and $\deg q(x) \geq \deg p(x) + 2$

(relative degree of $f$ is greater than or equal to 2)



**example:** show that

$$\int_{C_R} f(z)dz = 0$$

as $R \to \infty$ where $C_R$ is the arc $z = Re^{j\theta}$, $0 \leq \theta \leq \pi$

- $f(z) = (z+2)/(z^3+1)$ (relative degree of $f$ is 2)

$$|z+2| \leq |z| + 2 = R + 2, \quad |z^3+1| \geq ||z^3| - 1| = |R^3 - 1|$$

hence, $|f(z)| \leq \frac{R+2}{R^3-1}$ and apply the modulus of the integral

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz| \leq \frac{R+2}{R^3-1} \cdot \pi R = \pi \cdot \frac{1+\frac{2}{R^2}}{R - \frac{1}{R^2}}$$

the upper bound tends to zero as $R \to \infty$

- $f(z) = 1/(z^2+2z+2)$

$$z^2+2z+2 = (z-(1+j))(z-(1-j)) \triangleq (z-z_0)(z-\bar{z}_0)$$

hence, $|z-z_0| \geq ||z|-|z_0|| = R - |1+j| = R-\sqrt{2}$ and similarly,

$|z-z_0| \geq ||z|-|\bar{z}_0|| = R-\sqrt{2}$

then it follows that

$$|z^2+2z+2| \geq (R-\sqrt{2})^2 \quad \Rightarrow \quad |f(z)| \leq \frac{1}{(R-\sqrt{2})^2}$$

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|dz \leq \frac{1}{(R-\sqrt{2})^2}\cdot \pi R = \frac{\pi}{(1-\frac{\sqrt{2}}{R})^2}$$

the upper bound tends to zero as $R \to \infty$



**example:** compute $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

- define $f(z) = \dfrac{1}{1+z^2}$ and create a contour $C = C_1 \cup C_R$ as on page 13-29
- relative degree of $f$ is 2, so $\int_{C_R} f(z)dz = 0$ as $R \to \infty$
- $f(z)$ has poles at $z = j$ and $z=-j$ (no poles on the real axis)
- only the pole $z=j$ lies in the upper half plane
- by the residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z)dz}_{=0 \text{ as } R\to\infty}$$

$$I \;=\; j2\pi \operatorname*{Res}_{z=j} f(z) \;=\; j2\pi \lim_{z\to j}(z-j)f(z) \;=\; \pi$$



**example:** compute

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$$

- define $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ and create $C = C_1\cup C_R$ as on page 13-29
- relative degree of $f$ is 2, so $\int_{C_R} f(z)dz = 0$ as $R\to\infty$
- $f(z)$ has poles at $z = \pm ja$ and $z = \pm jb$ (no poles on the real axis)
- only the poles $z = ja$ and $z = jb$ lie in the upper half plane
- by the residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z)dz}_{=0 \text{ as } R\to\infty}$$

$$I = j2\pi\left[\operatorname*{Res}_{z=ja} f(z) + \operatorname*{Res}_{z=jb} f(z)\right] = j2\pi\left[\frac{a}{j2(a^2-b^2)} + \frac{b}{j2(b^2-a^2)}\right] = \frac{\pi}{a+b}$$

## Improper integrals from Fourier analysis

we can use residue theory to evaluate improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin mx\, dx, \quad \int_{-\infty}^{\infty} f(x)\cos mx\, dx, \quad \text{or} \quad \int_{-\infty}^{\infty} e^{jmx} f(x)\, dx$$

we note that $e^{jmz}$ is analytic everywhere, moreover

$$|e^{jmz}| = e^{jm(x+jy)} = e^{-my} < 1 \quad \text{for all } y \text{ in the upper half plane}$$

therefore, if $|f(z)| \leq M/R^k$ with $k > 1$, then so is $|e^{jmz} f(z)|$

hence, if $f(z)$ satisfies the conditions in page 13-30 then

$$\int_{-\infty}^{\infty} e^{jmx} f(x) dx = j2\pi \left[ \begin{array}{l} \text{sum of the residues of } e^{jmz} f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$



denote

$$S = \left[ \begin{array}{l} \text{sum of the residues of } e^{jmz} f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$

and note that $S$ can be *complex*

by comparing the real and imaginary part of the integral

$$\int_{-\infty}^{\infty} e^{jmx} f(x) dx = \int_{-\infty}^{\infty} (\cos mx + j\sin mx) f(x) dx = j2\pi S$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} \cos mx f(x)\, dx &= \operatorname{Re}(j2\pi S) = -2\pi \cdot \operatorname{Im} S \\ \int_{-\infty}^{\infty} \sin mx f(x)\, dx &= \operatorname{Im}(j2\pi S) = 2\pi \cdot \operatorname{Re} S \end{aligned}$$



**example:** compute $I = \int_{-\infty}^{\infty} \frac{\cos mx\, dx}{1+x^2}$

- define $f(z) = \frac{e^{jmz}}{1+z^2}$ and create $C = C_1 \cup C_R$ as on page 13-29
- relative degree of $f$ is $2$, so $\int_{C_R} f(z)dz = 0$ as $R \to \infty$
- $f$ has poles at $z = j$ and $z = -j$ (no poles on the real axis)
- the pole $z = j$ lies in the upper half plane
- by residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z) dz = \underbrace{\int_{-R}^{R} f(x) dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z) dz}_{=0 \text{ as } R\to\infty}$$

- therefore,

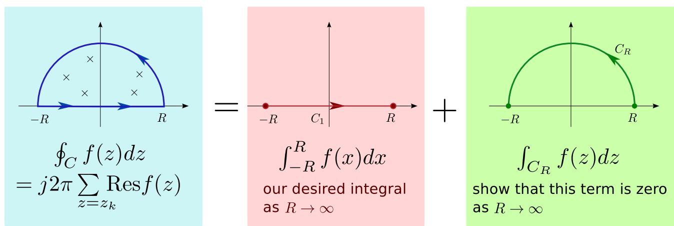
$$\int_{-\infty}^{\infty} \frac{e^{jmx}}{1+x^2} dx = j2\pi \operatorname*{Res}_{z=j} \frac{e^{jmz}}{1+z^2}$$
$$= j2\pi \lim_{z\to j} \frac{(z-j)e^{jmz}}{1+z^2} = \pi e^{-m}$$

- our desired integral can be obtained by

$$\int_{-\infty}^{\infty} \frac{\cos mx \, dx}{1+x^2} = \operatorname{Re}(\pi e^{-m}) = \pi e^{-m},$$
$$\int_{-\infty}^{\infty} \frac{\sin mx \, dx}{1+x^2} = \operatorname{Im}(\pi e^{-m}) = 0$$



## Summary of improper integrals




the examples of $f$ we have seen so far are in the form of

$$f(x) = \frac{p(x)}{q(x)}$$

where $p, q$ are polynomials and $\deg p(x) \geq \deg q(x) + 2$



the assumption on the degrees of $p, q$ is *sufficient* to guarantee that

$$\int_{C_R} f(z)e^{jaz} dz = 0 \quad (a > 0)$$

as $R \to \infty$ where $C_R$ is the arc $z = Re^{j\theta}, \ 0 \leq \theta \leq \pi$

we can relax this assumption to consider function $f$ such as

$$\frac{z}{z^2+2z+2}, \quad \frac{1}{z+1} \qquad \text{(relative degree is 1)}$$

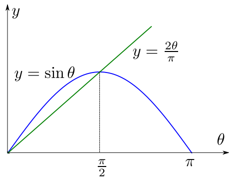
and obtain the same result by making use of **Jordan's inequality**

## Jordan inequality

for $R > 0$,

$$\int_0^{\pi} e^{-R\sin\theta} d\theta \; < \; \frac{\pi}{R}$$

Proof.




$$\begin{aligned} \sin\theta \;&\geq\; 2\theta/\pi, \quad 0 \leq \theta \leq \frac{\pi}{2} \\ e^{-R\sin\theta} \;&\leq\; e^{-2R\theta/\pi}, \quad R > 0, \;\; 0 \leq \theta \leq \frac{\pi}{2} \\ \int_0^{\pi/2} e^{-R\sin\theta} d\theta \;&\leq\; \frac{\pi}{2R} \end{aligned}$$

the last line is another form of the Jordan inequality

because the graph of $y = \sin\theta$ is symmetric about the line $\theta = \pi/2$



**example:** let $f(z) = \dfrac{z}{z^2 + 2z + 2}$ show that $\int_{C_R} f(z)e^{jaz}dz = 0$ for $a > 0$ as $R \to \infty$

- first note that $|e^{jaz}| = |e^{ja(x+jy)}| = |e^{jax} \cdot e^{-ay}| = e^{-ay} < 1$ (since $a > 0$)
- similar to page 13-34, we see that $|f(z)| \leq R/(R - \sqrt{2})^2 \triangleq M_R$ and

$$\left| \int_{C_R} f(z)e^{jaz}dz \right| \leq \int_{C_R} \frac{R}{(R-\sqrt{2})^2} \cdot \pi R = \frac{\pi}{(1 - \frac{\sqrt{2}}{R})^2}$$

  which **does not** tend to zero as $R \to \infty$

- however, for $z$ that lies on $C_R$, *i.e.*, $z = Re^{j\theta}$

$$f(z)e^{jaz} = f(z)e^{jaRe^{j\theta}} = f(z)e^{jaR(\cos\theta + j\sin\theta)} = f(z)e^{-aR\sin\theta} \cdot e^{jaR\cos\theta}$$



- if we find an upper bound of the integral, **and** use **Jordan's inequality**:

$$\begin{aligned} \left| \int_{C_R} f(z)e^{jaz}dz \right| &= \left| \int_0^{\pi} f(z)e^{-aR\sin\theta} \cdot e^{jaR\cos\theta} jRe^{j\theta} d\theta \right| \\ &\leq \int_0^{\pi} \left| f(z)e^{-aR\sin\theta} \cdot e^{jaR\cos\theta} jRe^{j\theta} \right| d\theta \\ &= RM_R \int_0^{\pi} e^{-aR\sin\theta} d\theta \\ &< \frac{\pi M_R}{a} \end{aligned}$$

  the final term approach $0$ as $R \to \infty$ because $M_R \to 0$

**conclusion:** then we can apply the residue's theorem to integrals like

$$\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2 + 2x + 2} dx$$

## Inversion of Laplace transforms

recall the definitions:

$$F(s) \triangleq \mathcal{L}[f(t)] \triangleq \int_0^\infty f(t)e^{-st}dt$$
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{j2\pi}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}ds$$

**Theorem:** suppose $F(s)$ is analytic everywhere except at the poles

$$p_1, p_2, \ldots, p_n,$$

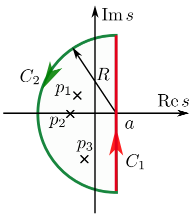
all of which lie to the **left** of the vertical line $\mathrm{Re}(s) = a$ (a convergence factor)

**if** $|F(s)| \leq M_R$ **and** $M_R \to 0$ as $s \to \infty$ through the half plane $\mathrm{Re}(s) \leq a$ then

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{n} \operatorname*{Res}_{s=p_i} F(s)e^{st}$$



**Proof sketch.**




parametrize $C_1$ and $C_2$ by

$$C_1 = \{z \mid z = a + jy, \quad -R \leq y \leq R\}$$
$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$$

1. create a huge semicircle that is large enough to contain all the poles of $F(s)$
2. apply the Cauchy's residue theorem to conclude that

$$\int_{C_1} e^{st}F(s)ds = j2\pi \sum_{k=1}^{n} \operatorname*{Res}_{s=p_k}[e^{st}F(s)] - \int_{C_2} e^{st}F(s)ds$$

3. prove that the integral along $C_2$ is zero when the circle radius goes to $\infty$



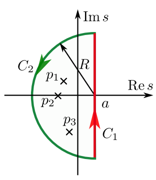
**choose** $a$ **and** $R$: choose the center and radius of the circle

- $a > 0$ is so large that all the poles of $F(s)$ lie to the left of $C_1$

$$a > \max_{k=1,2,\ldots,n} \mathrm{Re}(p_k)$$

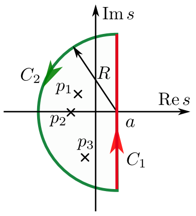
- $R > 0$ is large enough so that all poles of $F(s)$ are enclosed by the semicircle
  if the maximum modulus of $p_1, p_2, \ldots, p_n$ is $R_0$ then

$$\forall k, \ |p_k - a| \leq |p_k| + a \leq R_0 + a \implies \text{pick } R > R_0 + a$$




$$C_1 = \{z \mid z = a + jy, \quad -R \leq y \leq R\}$$
$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$$

**integral along $C_2$ is zero**




$$C_1 = \{z \mid z = a + jy, \quad -R \le y \le R\}$$
$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$$

- for $s = a + Re^{j\theta}$ and $ds = jRe^{j\theta}d\theta$, the integral becomes

$$\left|\int_{C_2} e^{st}F(s)ds\right| = \left|\int_{\pi/2}^{3\pi/2} e^{at}\cdot e^{Rt\cos\theta + jRt\sin\theta}F(a+Re^{j\theta})Rje^{j\theta}d\theta\right|$$



- apply the modolus of the integral

$$\left|\int_{C_2} e^{st}F(s)ds\right| \le \int_{\pi/2}^{3\pi/2} \left|e^{at}e^{Rt\cos\theta}\cdot e^{jRt\sin\theta}F(a+Re^{j\theta})Rje^{j\theta}\right|\ d\theta$$

- since $|F(s)| \le M_R$ for $s$ that lies on $C_2$

$$\left|\int_{C_2} e^{st}F(s)ds\right| \le M_R R e^{at}\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\ d\theta$$

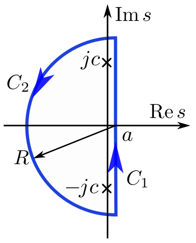
- make change of variable $\phi = \theta - \pi/2$ and apply the **Jordan inequality**

$$\left|\int_{C_2} e^{st}F(s)ds\right| \le M_R R e^{at}\underbrace{\int_0^{\pi} e^{-Rt\sin\phi}\ d\phi}_{<\pi/Rt} \quad < \quad \frac{\pi M_R e^{at}}{t}$$

the last term approaches zero as $R \to \infty$ because $M_R \to 0$ (by assumption)



**example:** find $\mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{s}{(s^2+c^2)^2}$ and $c > 0$




$C_2 = \{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$
poles of $F(s)$ are $s = \pm jc$ so we choose $a > 0$
the semicircle must enclose all the pole
so we have $R > a + c$

first we verifty that $|F(s)| \le M_R$ and $M_R \to 0$ as $s \to \infty$ for $s$ on $C_2$

we note that $|s| = |a + Re^{j\theta}| \le a + R$ and $|s| \ge |a - R| = R - a$

since $|s^2 + c^2| \ge ||s|^2 - c^2| \ge (R-a)^2 - c^2 > 0$, then

$$|F(s)| = \frac{|s|}{|s^2+c^2|^2} \le \frac{(R+a)}{[(R-a)^2-c^2]^2} \triangleq M_R \to 0 \quad \text{as } R \to \infty$$

therefore, we can apply the theorem on page 13-46

$$\mathcal{L}^{-1}[F(s)] = \sum \operatorname*{Res}_{s=s_k}[e^{st}F(s)] = \operatorname*{Res}_{s=jc} \frac{se^{st}}{(s^2+c^2)^2} + \operatorname*{Res}_{s=-jc} \frac{se^{st}}{(s^2+c^2)^2}$$

poles of $F(s)$ are $s = \pm jc$ (double poles)

$$\begin{aligned}
\operatorname*{Res}_{s=jc} e^{st}F(s) &= \lim_{s\to jc} \frac{d}{ds}\left[\frac{se^{st}}{(s+jc)^2}\right] = \left[\frac{e^{st}(1+ts)}{(s+jc)^2} - \frac{2se^{st}}{(s+jc)^3}\right]_{s=jc} \\
&= \frac{te^{jct}}{j4c} \\
\operatorname*{Res}_{s=-jc} e^{st}F(s) &= \lim_{s\to -jc} \frac{d}{ds}\left[\frac{se^{st}}{(s-jc)^2}\right] = \left[\frac{e^{st}(1+ts)}{(s-jc)^2} - \frac{2se^{st}}{(s-jc)^3}\right]_{s=-jc} \\
&= -\frac{te^{-jct}}{j4c}
\end{aligned}$$

hence $\mathcal{L}^{-1}[F(s)] = \frac{t}{4jc}(e^{jct} - e^{-jct}) = \dfrac{t\sin ct}{2c}$



**example:** find $\mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{1}{(s+a)^2+b^2}$

$F(s)$ has poles at $s = -a \pm jb$ (simple poles)

$$\mathcal{L}^{-1}[F(s)] = \operatorname*{Res}_{s=-a+jb} e^{st}F(s) + \operatorname*{Res}_{s=-a-jb} e^{st}F(s)$$

(provided that $|F(s)| \leq M_R$ and $M_R \to 0$ as $s \to \infty$ on $C_2$ ... please check ✎)

$$\begin{aligned}
\operatorname*{Res}_{s=-a+jb} &= \lim_{s=-a+jb} \frac{e^{st}}{s+a+jb} = \frac{e^{(-a+jb)t}}{j2b} \\
\operatorname*{Res}_{s=-a-jb} &= \lim_{s=-a-jb} \frac{e^{st}}{s+a-jb} = \frac{e^{(-a-jb)t}}{-j2b}
\end{aligned}$$

hence, $\mathcal{L}^{-1}[F(s)] = \dfrac{e^{-at}(e^{jbt} - e^{-jbt})}{2jb} = \dfrac{e^{-at}\sin(bt)}{b}$



## References


Chapter 6-7 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 7 in

T. W. Gamelin, *Complex Analysis*, Springer, 2001

Chapter 22 in

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## Exercises

1. Find the residues of
$$f(z) = \frac{z^2+2}{(z+2)^2(z^2+4)}$$
at all its poles in the finite plane.

2. Consider $f(z) = e^z/\sin^2 z$. Show that $z = \pi$ is a pole of order $2$ (double pole) and find the residue of $f$ at $z = \pi$.

3. Explain how to evaluate the integral
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)}\,dx$$
by applying the residue theorem, and find the value of the integral.

4. Evaluate the integral
$$\int_0^{2\pi} \frac{2\cos 2\theta}{5-4\cos\theta}\,d\theta$$
by applying the residue theorem.

5. Explain how to evaluate
$$\int_{-\infty}^{\infty} \frac{\cos(\pi m x)}{(x^2+1)^2}\,dx$$
by applying the residue theorem and compute the integral.

6. Find the inverse Laplace transform of
$$F(s) = \frac{s^2-2s-23}{(s+2)(s^2+6s+13)}$$
by using the residue theorem. State the assumptions required by your calculation and show that those assumptions hold in this problem.

# เอกสารอ้างอิง


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