8. Complex Numbers

- sums and products
- basic algebraic properties
- complex conjugates
- exponential form
- principal arguments
- roots of complex numbers
- regions in the complex plane

Introduction

we denote a complex number z by

$$z = x + jy$$

where

- $x = \mathrm{Re}(z)$ (real part of z)
- $y = \mathrm{Im}(z)$ (imaginary part of z)
- $j = \sqrt{-1}$

Sum and Product

consider two complex numbers

$$z_1 = x_1 + jy_1, \quad z_2 = x_2 + jy_2$$

the sum and product of two complex number are defined as:

- $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ addition
- $z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(y_1 x_2 + x_1 y_2)$ multiplication

example:

$$(-3 + j5)(1 - 2j) = 7 + j11$$

Algebraic properties ✎

- $z_1 = z_2 \iff \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$ — equality
- $z_1 + z_2 = z_2 + z_1$ — commutative
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ — associative
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ — distributive
- $-z = -x - jy$ — addtive inverse
- $z^{-1} = \dfrac{x}{x^2 + y^2} - j\dfrac{y}{x^2 + y^2}$ — multiplicative inverse

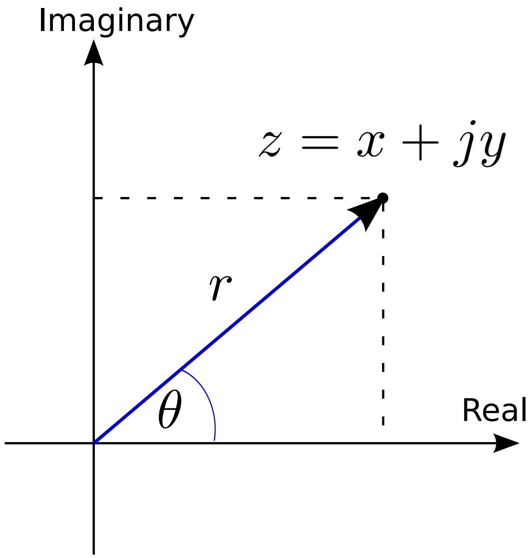
Complex conjugate and Moduli ✎

modulus (or absolute value): $|z| = \sqrt{x^2 + y^2}$

complex conjugate: $\bar{z} = x - jy$

- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ triangle inequality
- $|z_1 + z_2| \geq ||z_1| - |z_2||$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\bar{z_2}}$, if $z_2 \neq 0$
- $\mathrm{Re}(z) = (z + \bar{z})/2$ and $\mathrm{Im}(z) = (z - \bar{z})/2j$

Argument of complex numbers



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = r(\cos\theta + j\sin\theta)$$

$$r = |z|$$

$$\theta = \tan^{-1}(y/x) \quad \triangleq \quad \arg z$$

(called an argument of z)

principal value of $\arg z$ denoted by $\operatorname{Arg} z$ is the unique θ such that $-\pi < \theta \leq \pi$

$$\arg z = \operatorname{Arg} z + 2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

example: $\operatorname{Arg}(-1 + j) = \frac{3\pi}{4}, \quad \arg z = \frac{3\pi}{4} + 2n\pi, \quad n = 0, \pm 1, \ldots$

Polar representation

Euler's formula ✌

$$e^{j\theta} = \cos\theta + j\sin\theta$$

a polar representation of $z = x + jy$ (where $z \neq 0$) is

$$z = re^{j\theta}$$

where $r = |z|$ and $\theta = \arg z$

example:

$$(-1+j) = \sqrt{2}\, e^{j3\pi/4} = \sqrt{2}\, e^{j(3\pi/4 + 2n\pi)}, \quad n = 0, \pm 1, \ldots$$

(there are infinite numbers of polar forms for $-1+j$)

let $z_1 = r_1\ e^{j\theta_1}$ and $z_2 = r_2\ e^{j\theta_2}$

properties ✎

- $z_1 z_2 = r_1 r_2\ e^{j(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\ e^{j(\theta_1-\theta_2)}$
- $z^{-1} = \frac{1}{r} e^{-j\theta}$
- $z^n = r^n\ e^{jn\theta}, \quad n = 0, \pm 1, \ldots$

de Moivre's formula ✎

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta, \quad n = 0, \pm 1, \pm 2, \ldots$$

example: prove the following trigonometric identity

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

from de Moivre's formula,

$$\begin{aligned}\cos 3\theta + j\sin 3\theta &= (\cos\theta + j\sin\theta)^3 \\ &= \cos^3\theta + j3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta\end{aligned}$$

and the identity is readily obtained from comparing the real part of both sides

Arguments of products

an argument of the product $z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$ is given by

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

example: $z_1 = -1$ and $z_2 = -1 + j$

$$\arg(z_1 z_2) = \arg(1 - j) = 7\pi/4, \quad \arg z_1 + \arg z_2 = \pi + 3\pi/4$$

this result is not always true if $\arg$ is replaced by Arg

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(1 - j) = -\pi/4, \quad \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \pi + 3\pi/4$$

✎ more properties of the argument function

- $\arg(\bar{z}) = -\arg z$
- $\arg(1/z) = -\arg z$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(no need to memorize these formulae)

Roots of complex numbers

an nth root of $z_0 = r_0 e^{j\theta_0}$ is a number $z = re^{j\theta}$ such that $z^n = z_0$, or

$$r^n e^{jn\theta} = r_0 e^{j\theta_0}$$

note: two nonzero complex numbers

$$z_1 = r_1 e^{j\theta_1} \quad \text{and} \quad z_2 = r_2 e^{j\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2k\pi$$

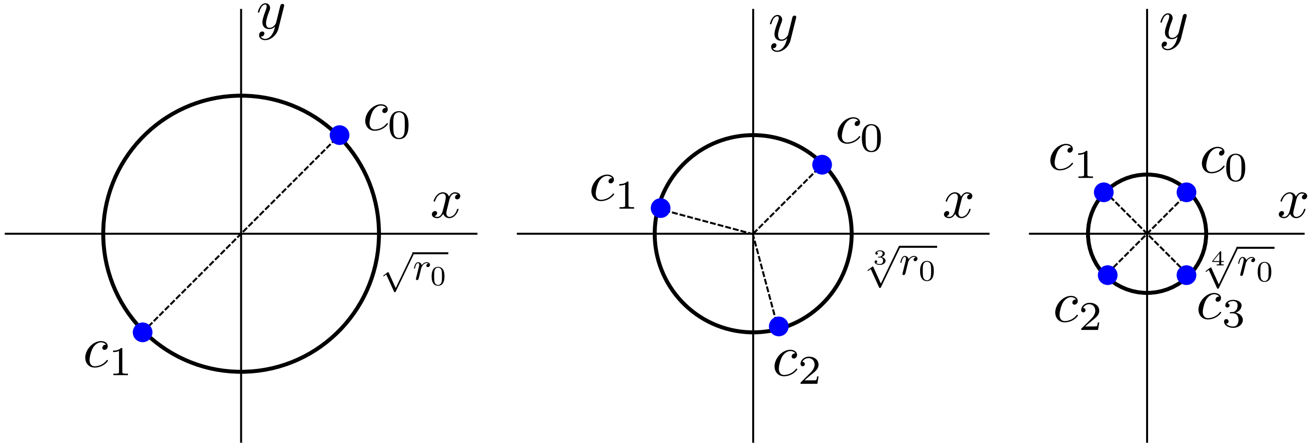
for some $k = 0, \pm 1, \pm 2, \ldots$

therefore, the nth roots of z_0 are

$$z = \sqrt[n]{r_0} \ \exp \left[j \left(\frac{\theta_0 + 2k\pi}{n} \right) \right] \quad k = 0, \pm 1, \pm 2, \ldots$$

all of the **distinct** roots are obtained by

$$c_k = \sqrt[n]{r_0} \ \exp \left[j \left(\frac{\theta_0 + 2k\pi}{n} \right) \right] \quad k = 0, 1, \ldots, n-1$$



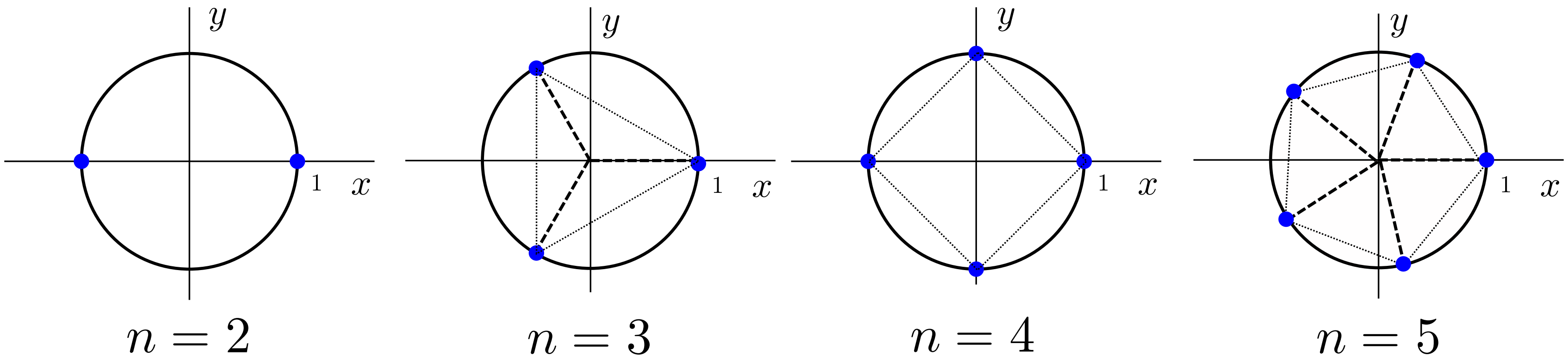
the roots lie on the circle $|z| = \sqrt[n]{r_0}$ and equally spaced every $2\pi/n$ rad when $-\pi < \theta_0 \leq \pi$, we say c_0 is the **principal root**

example 1: find the n roots of 1 for $n = 2, 3, 4$ and 5

$$1 = 1 \cdot \exp\,[j(0 + 2k\pi)], \quad k = 0, \pm 1, \pm 2, \ldots$$

the distinct n roots of 1 are

$$c_k = \sqrt[n]{r_0}\ \exp\left[j\left(\frac{0 + 2k\pi}{n}\right)\right] \quad k = 0, 1, \ldots, n-1$$

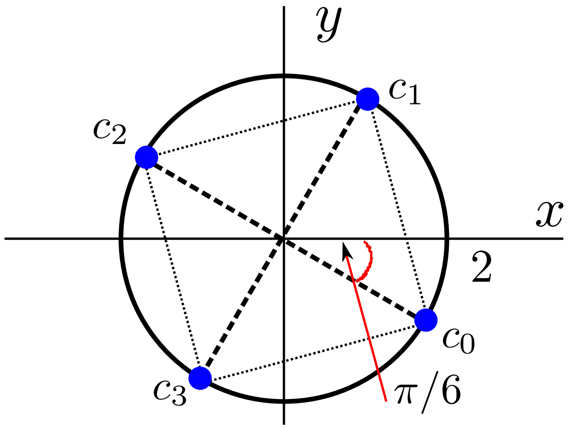


example 2: find $(-8 - j8\sqrt{3})^{1/4}$

write $z_0 = -8 - j8\sqrt{3} = 16e^{j(-\pi+\pi/3)} = 16e^{j(-2\pi/3)}$

the four roots of z_0 are

$$c_k = (16)^{1/4}\ \text{exp}\ \left[j\left(\frac{-2\pi/3 + 2k\pi}{4}\right)\right] \quad k = 0, 1, 2, 3$$

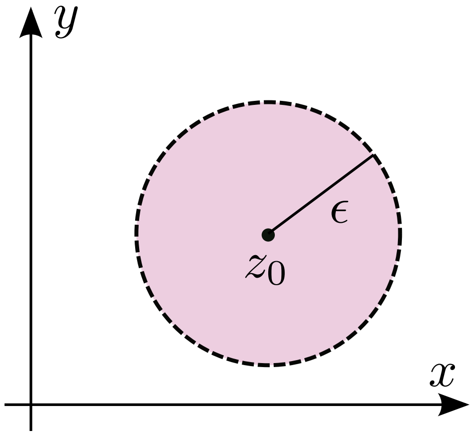


$$\begin{aligned}
c_0 &= 2e^{j(-2\pi/12)} &&= 2e^{-j\pi/6} &&= \sqrt{3} - j \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+2\pi}{4}\right)} &&= 2e^{j\pi/3} &&= 1 + j\sqrt{3} \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+4\pi}{4}\right)} &&= 2e^{j5\pi/6} &&= -\sqrt{3} + j \\
c_1 &= 2e^{j\left(\frac{-2\pi/3+6\pi}{4}\right)} &&= 2e^{j4\pi/3} &&= -1 - j\sqrt{3}
\end{aligned}$$

Regions in the Complex Plane

- interior, exterior, boundary points
- open and closed sets
- loci on the complex plane

Regions in the complex plane



an ϵ **neighborhood** of z_0 is the set

$$\{z \in \mathbf{C} \mid |z - z_0| < \epsilon\}$$

Definition: a point z_0 is said to be

- an **interior point** of a set S if there exists a neighborhood of z_0 that contains *only points* of S
- an **exterior point** of S when there exists a neighborhood of it containing *no points* of S
- a **boundary point** of S if it is neither an interior nor an exterior point of S

the **boundary** of S is the set of *all* boundary points of S

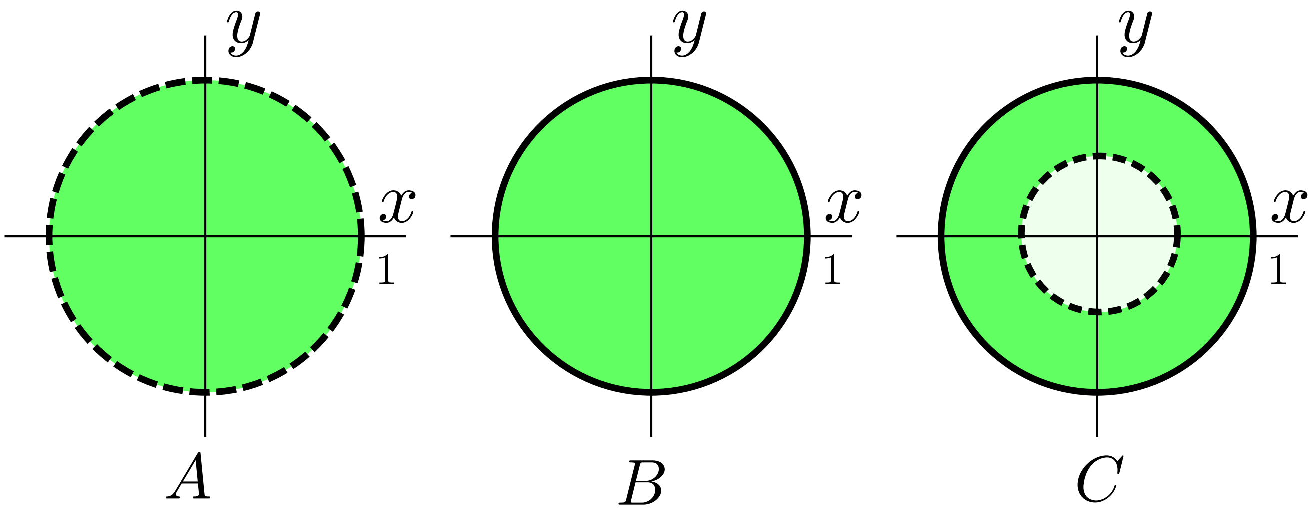
examples on the real axis: $S_1 = (0, 1)$, $S_2 = [0, 1]$, and $S_3 = (0, 1]$

in *real analysis*, an ϵ neighborhood of $x_0 \in \mathbf{R}$ is the set

$$\{x \in \mathbf{R} \mid |x - x_0| < \epsilon \}$$

- any $x \in (0, 1)$ is an interior point of S_1, S_2, and S_3
- any $x \in (-\infty, 0) \cup (1, \infty)$ is an exterior point of S_1, S_2 and S_3
- 0 and 1 are boundary points of S_1, S_2 and S_3

examples on the complex plane:



- any point $z \in \mathbf{C}$ with $|z| < 1$ is an interior point of A and B
- any point $z \in \mathbf{C}$ with $1/2 < |z| < 1$ is an interior point of C
- any point $z \in \mathbf{C}$ with $|z| > 1$ is an exterior point of A and B
- any point $z \in \mathbf{C}$ with $0 < |z| < 1/2$ or $|z| > 1$ is an exterior point of C
- the circle $|z| = 1$ is the boundary of A and B
- the union of the circles $|z| = 1$ and $|z| = 1/2$ is the boundary of C

Open and Closed sets

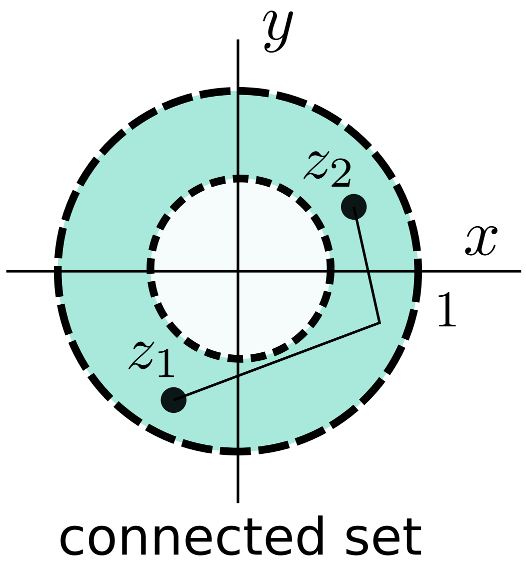
- a set is **open** if and only if each of its points is an interior point
- a set is **closed** if it contains all of its boundary points
- the **closure** of a set S is the *closed* set consisting of all points in S together with the boundary of S
- some sets are neither open nor closed

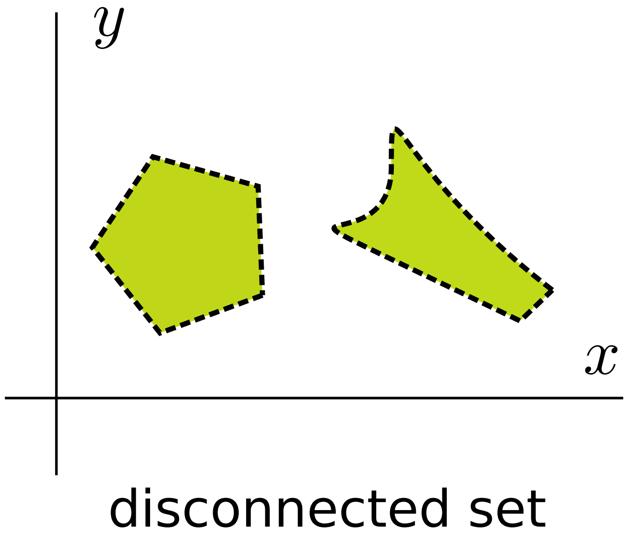
from the examples on page 8-18 and page 8-19,

- S_1 is open, S_2 is closed, S_3 is neither open nor closed
- S_2 is the closure of S_1
- A is open, B is closed, C is neither open nor closed
- B is the closure of A

Connected sets

an open set S is said to be **connected** if any pair of points z_1 and z_2 in S can be joined by a *polygonal line* that lies entirely in S





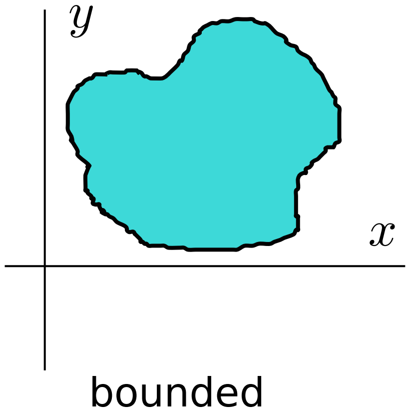
- a nonempty open set that is connected is called a **domain**
- any neighborhood is a domain
- a domain with some, none, or all of its boundary points is called a **region**

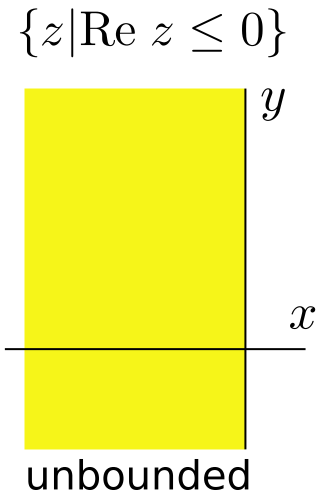
Bounded sets

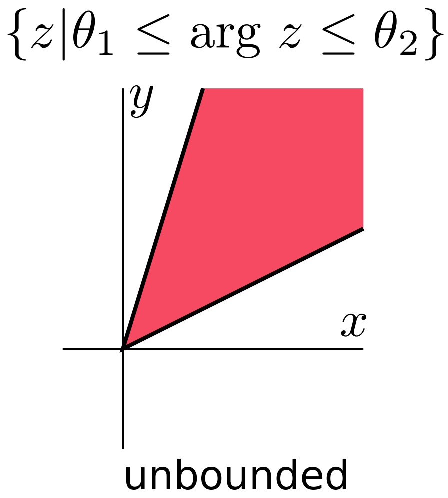
a set S is said to be **bounded** if for any point $z \in S$,

$$|z| \leq M, \quad \text{for some } M < \infty$$

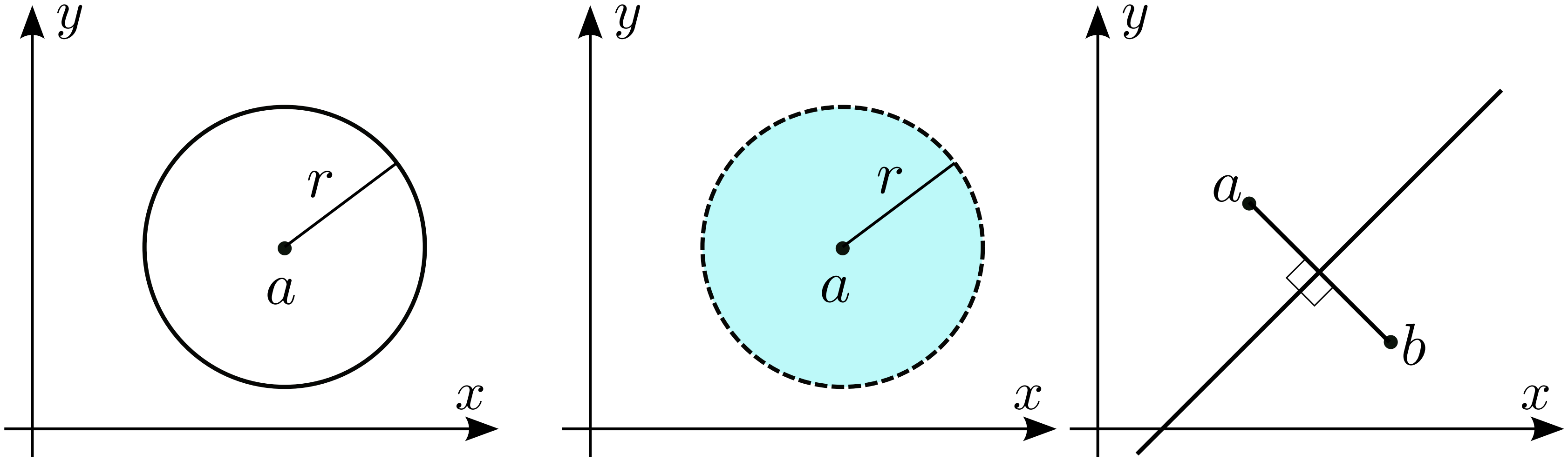
otherwise it is **unbounded**







Loci in the complex plane



- $|z - a| = r$, $a \in \mathbf{C}, r \in \mathbf{R}$

- $|z - a| < r$, $a \in \mathbf{C}, r \in \mathbf{R}$

- $|z - a| = |z - b|$, $a, b \in \mathbf{C}$

References

Chapter 1 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009