10. Elementary Functions

- exponential function
- logarithmic function
- complex components
- trigonometric function
- hyperbolic functions
- branches for multi-valued functions

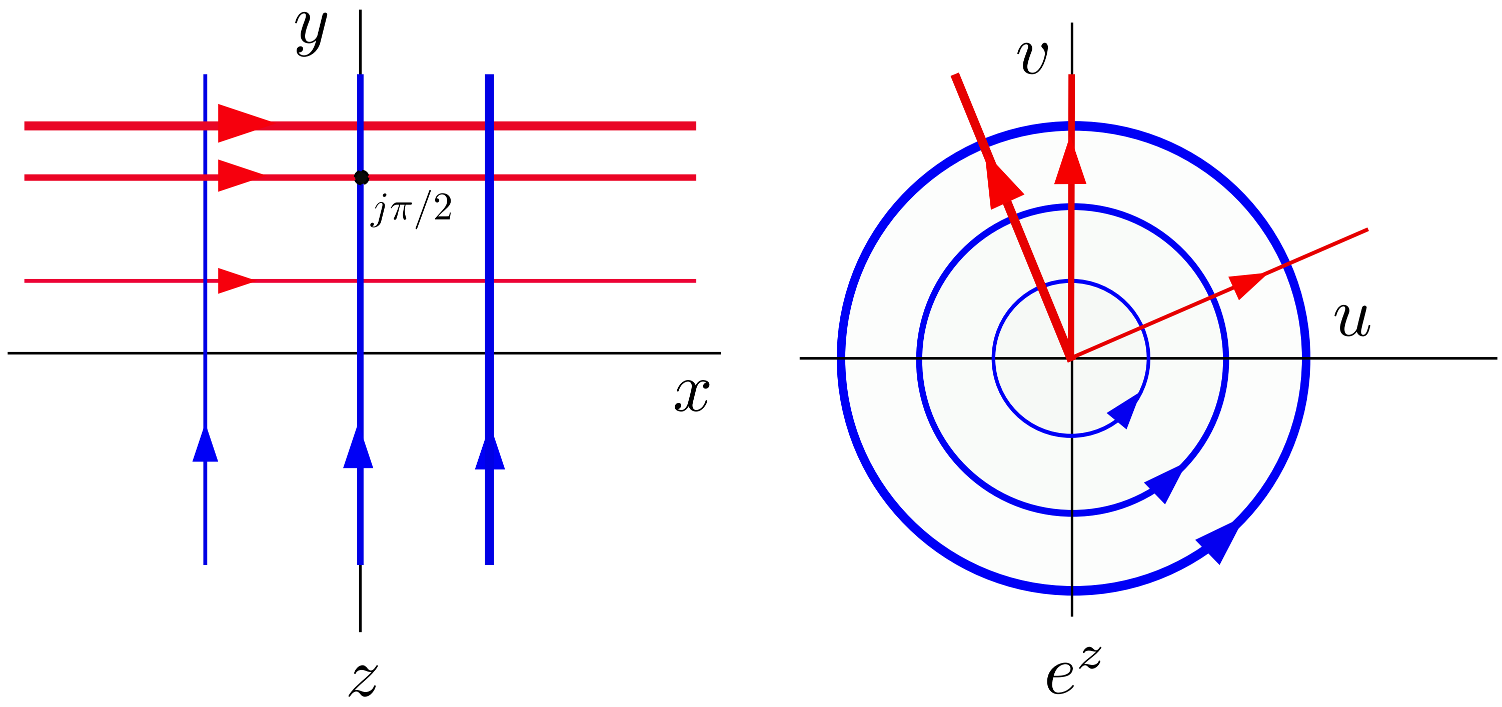
Exponential function

from $z = x + jy$, an exponential function is defined as

$$f(z) = e^z \quad = \quad e^x \cos y + je^x \sin y$$

Properties ✎

- $f(z) = e^z$ is analytic everywhere on **C** (e^z is entire)
- $f'(z) = e^z$
- $e^{z+w} = e^z e^w \,, \quad z, w \in \mathbf{C}$ (addition formula)
- $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$ (so $e^z \neq 0$)
- if z is pure imaginary, then e^z is periodic



images under $f(z) = e^z$

- images of horizontal lines are rays pointing from the origin
- images of vertical lines are circles centered at the origin

Logarithmic function

the definition of $\log$ function is based on solving

$$e^w = z$$

for w where z is any **nonzero** complex number, and we call $w = \log z$

write $z = re^{j\Theta}(-\pi < \Theta \leq \pi)$ and $w = u + jv$, so we have

$$e^u = r, \qquad v = \Theta + 2n\pi$$

thus the definition of the (multiple-valued) **logarithmic** function of z is

$$\log z = \log r + i(\Theta + 2n\pi), \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

if only the principle value of $\arg z$ is used ($n = 0$), then $\log z$ is *single-valued* the **principal value** of $\log z$ is defined as

$$\operatorname{Log} z = \log r + i \operatorname{Arg} z$$

where $r = |z|$ and recall that $\operatorname{Arg} z$ is the principal argument of z

- $\operatorname{Log} z$ is single-valued
- $\log z = \operatorname{Log} z + j2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots)$

note: when z is complex, one should **not** jump into the conclusion that

$$\log(e^z) = z \qquad (\log \text{ is multiple-valued})$$

instead, if $z = x + jy$, we should write

$$\begin{aligned}\log(e^z) &= \log|e^z| + j(\operatorname{Arg}(e^z) + 2n\pi) = \log|e^x| + j(y + 2n\pi) \\ &= z + j2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)\end{aligned}$$

example: find $\log z$ for $z = -1 + j$, $z = 1$, and $z = -1$

- if $z = -1 + j$ then $r = \sqrt{2}$ and $\operatorname{Arg} z = 3\pi/4$

$$\begin{aligned} \log z &= \log \sqrt{2} + j(3\pi/4 + 2n\pi), \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \operatorname{Log} z &= \log \sqrt{2} + j3\pi/4 \end{aligned}$$

- if $z = 1$ then $r = 1$ and $\operatorname{Arg} z = 0$

$$\begin{aligned} \log z &= 0 + j2n\pi = j2n\pi, \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \operatorname{Log} z &= 0 \qquad \text{(as expected)} \end{aligned}$$

- if $z = -1$ then $r = 1$ and $\operatorname{Arg} z = \pi$

$$\begin{aligned} \log z &= \log 1 + j(\pi + 2n\pi) = j(2n+1)\pi, \quad (n = 0, \pm 1, \pm 2, \ldots) \\ \operatorname{Log} z &= j\pi = j \qquad \text{(now we can find } \log \text{ of a negative number)} \end{aligned}$$

Complex exponents

let $z \neq 0$ and c be any complex number, the function z^c is defined via

$$z^c = e^{c \log z}$$

where $\log z$ is the multiple-valued logarithmic function

let Θ be the principal value of $\arg z$ and let $c = a + jb$

$$z^c = e^{c \log z} = e^{(a+jb)(\log|z| + j(\Theta + 2n\pi))}, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

example: find j^j

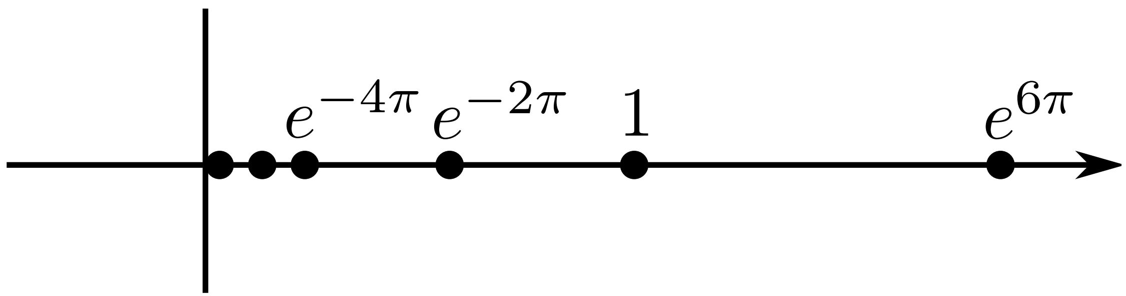
$$j^j = e^{j(\log j)} = e^{j(\log 1 + j(\pi/2 + 2n\pi))} = e^{-(1/2+2n)\pi}, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

(complex power of a complex can become real numbers) more example: the values of j^{-j} are given by

$$j^{-j} = e^{-j(\log j)} = e^{-j(\log 1 + j(\pi/2 + 2m\pi))} = e^{(1/2+2m)\pi}, \quad (m = 0, \pm 1, \pm 2, \ldots)$$

if we multiply the values of j^j by those of j^{-j} we obtain infinitely many values of

$$e^{2k\pi}, \quad -\infty < k < \infty$$



thus, the usual algebraic rules *do not* apply to z^c when they are multi-valued !

$$(j^j) \cdot (j^{-j}) \neq j^0 = 1$$

Trigonometric functions

by using Euler's formula

$$e^{jx} = \cos x + j \sin x, \qquad \text{for any } \textit{real} \text{ number } x$$

we can write

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}, \qquad \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

hence, it is *natural* to define trigonometric functions of a complex number z as

$$\begin{aligned} \sin z &= \frac{e^{jz} - e^{-jz}}{2j}, & \cos z &= \frac{e^{jz} + e^{-jz}}{2}, & \tan z &= \frac{\sin z}{\cos z} \\ \csc z &= \frac{1}{\sin z}, & \sec z &= \frac{1}{\cos z}, & \cot z &= \frac{1}{\tan z} \end{aligned}$$

Properties ✎

- $\sin z$ and $\cos z$ are entire functions (since e^{jz} and e^{-jz} are entire)
- $\dfrac{d}{dz}\sin z = \cos z$ and $\dfrac{d}{dz}\cos z = -\sin z$ (use $\frac{d}{dz}e^{jz} = je^{jz}$)
- $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$ (sine is odd and cosine is even)
- $\sin(z + 2\pi) = \sin z$ and $\sin(z + \pi) = -\sin z$
- $\cos(z + 2\pi) = \cos z$ and $\cos(z + \pi) = -\cos z$
- $\sin(z + \pi/2) = \cos z$ and $\sin(z - \pi/2) = -\cos z$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$

Hyperbolic functions

the hyperbolic sine, cosine, and tangent of a complex number are defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}, \qquad \tanh z = \frac{\sinh z}{\cosh z}$$

(as they are with a real variable)

Properties ✎

- $\sinh z$ and $\cosh z$ are entire (since e^z and e^{-z} are entire)
- $\tanh z$ is analytic in every domain in which $\cosh z \neq 0$
- $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$
- $\frac{d}{dz}\tanh z = \text{sech}^2 z$

Branches for multiple-valued functions

we often need to investigate the differentiability of a function f

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

what happen if f is multiple-valued (like $\arg z$, z^c) ?

- have to make sure if the two funtion values tend to the same value in the limit
- have to choose one of the function values in a consistent way

restricting the values of a multiple-valued functions to make it single-valued in some region is called choosing **a branch** of the function

a **branch** of f is any single-valued function F that is analytic in some domain

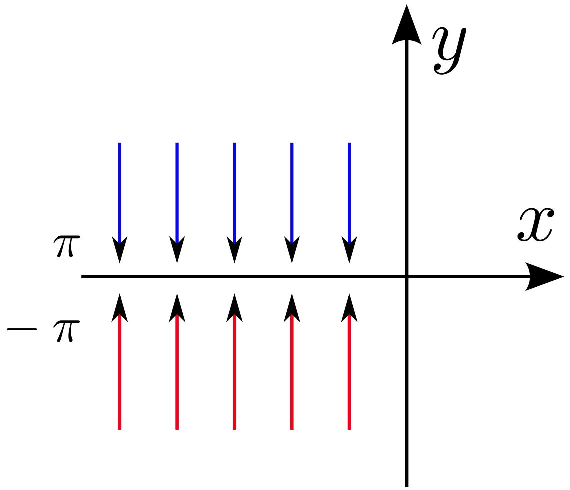
Branches for logarithmic functions

we define the **principal branch** Log of the lograrithmic function as

$$\mathrm{Log}\, z = \mathrm{Log}\,|z| + j\,\mathrm{Arg}(z), \quad -\pi < \mathrm{Arg}(z) < \pi$$

where $\mathrm{Arg}(z)$ is the principle value of $\arg(z)$

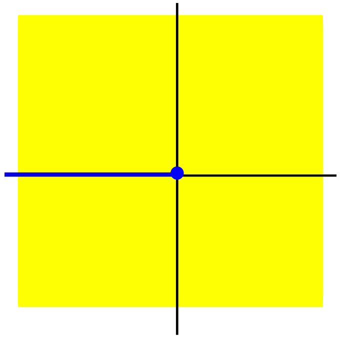
- $\mathrm{Log}\, z$ is single-valued
- $\mathrm{Log}\, z$ is *not continuous* along **the negative real axis** (because of $\mathrm{Arg}\, z$)



$$z = -x + j\epsilon \quad \Longrightarrow \quad \mathrm{Arg}\, z \to \pi \quad \text{as } \epsilon \to 0$$

$$z = -x - j\epsilon \quad \Longrightarrow \quad \mathrm{Arg}\, z \to -\pi \quad \text{as } \epsilon \to 0$$

a **branch cut** is portion of curve that is introduced to define a branch F



$D = \mathbf{C} \backslash (-\infty, 0]$

- points on the branch cut for f are singular points
- the negative real axis is a **branch cut** for the Log function
- $\mathrm{Log}\, z$ is single-valued and continuous in $D = \mathbf{C} \backslash (-\infty, 0]$

let z_0 be any point in D and $w_0 = \mathrm{Log}\, z_0$ (or $z_0 = e^{w_0}$)

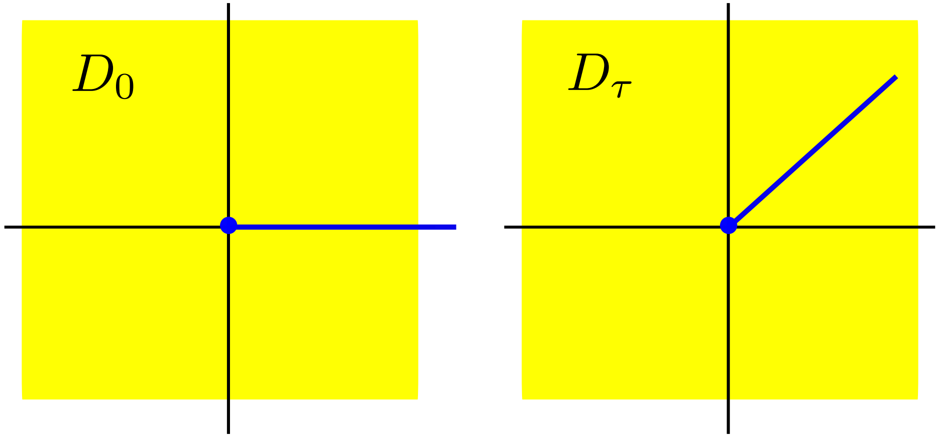
$$\frac{d}{dz} \mathrm{Log}\, z_0 = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{z \to z_0} \frac{1}{\frac{z - z_0}{w - w_0}} = \lim_{w \to w_0} \frac{1}{\frac{z - z_0}{w - w_0}}$$

$$= \lim_{w \to w_0} \frac{1}{\frac{e^w - e^{w_0}}{w - w_0}} = \frac{1}{\frac{d}{dz} e^w \Big|_{w = w_0}} = \frac{1}{e^{w_0}} = \frac{1}{z_0}$$

(we have used single-valuedness and continuity of the Log function)

$\mathrm{Log}\, z$ is analytic in D

other branches of $\log z$



$$
\begin{aligned}
\log(z) &= \operatorname{Log}|z| + j\arg(z), \quad 0 < \arg(z) < 2\pi \\
\log(z) &= \operatorname{Log}|z| + j\arg_\tau(z), \quad \tau < \arg_\tau(z) < \tau + 2\pi
\end{aligned}
$$

a branch $\log(z)$ is analytic everywhere on D_τ

any point that is common to all branch cuts of f is called a **branch point**

the origin is a **branch point** of the log function

example: suppose we have to compute the derivative of

$$f(z) = \log(z^2 - 1) \quad \text{at point } z = j$$

choose a branch of f which is analytic in a region containing the point

$$j^2 - 1 = -2$$

the principal branch is not analytic there, so we choose another branch

e.g., choose $\log(z) = \operatorname{Log}|z| + j\arg(z)$, $0 < \arg z < 2\pi$; then by chain rule,

$$f'(j) = \left.\frac{2z}{z^2 - 1}\right|_{z=j} = \frac{2j}{j^2 - 1} = -j$$

Branches for the complex power

define the **principal branch** of z^c to be

$$e^{c \operatorname{Log} z}$$

where $\operatorname{Log} z$ is the principal branch of $\log z$

- since the exponential function is entire, the principal branch of z^c is analytic in D where $\operatorname{Log} z$ is analytic
- using the chain rule

$$\frac{d}{dz}\left(e^{c \operatorname{Log} z}\right)\Big|_{z=z_0} = e^{c \operatorname{Log} z_0} \frac{c}{z_0} = \frac{c z_0^c}{z_0} = c z_0^{c-1}$$

  provided that we use the same branch of z^c on both sides of the equation

($z^a z^b = z^{a+b}$ iff we use the same branch for the complex power on both sides)

References

Chapter 3 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 1 in

T. W. Gameline, *Complex Analysis*, Springer, 2001

J. F. O'Farrill, *Lecture note on Complex Analysis*, University of Edinburgh, `http://www.maths.ed.ac.uk/ jmf/Teaching/MT3/ComplexAnalysis.pdf`