8. Terminology in Random Processes

- definition and specification of RPs
- statistics: pdf, cdf, mean, variance
- statistical properties: independence, correlation, orthogonal, stationarity

Definition

elements to be considered:

- let Θ be a random variable (that its outcome, θ is mapped from a sample space S)
- let t be a deterministic value (referred to as 'time') and $t \in T$

definition:

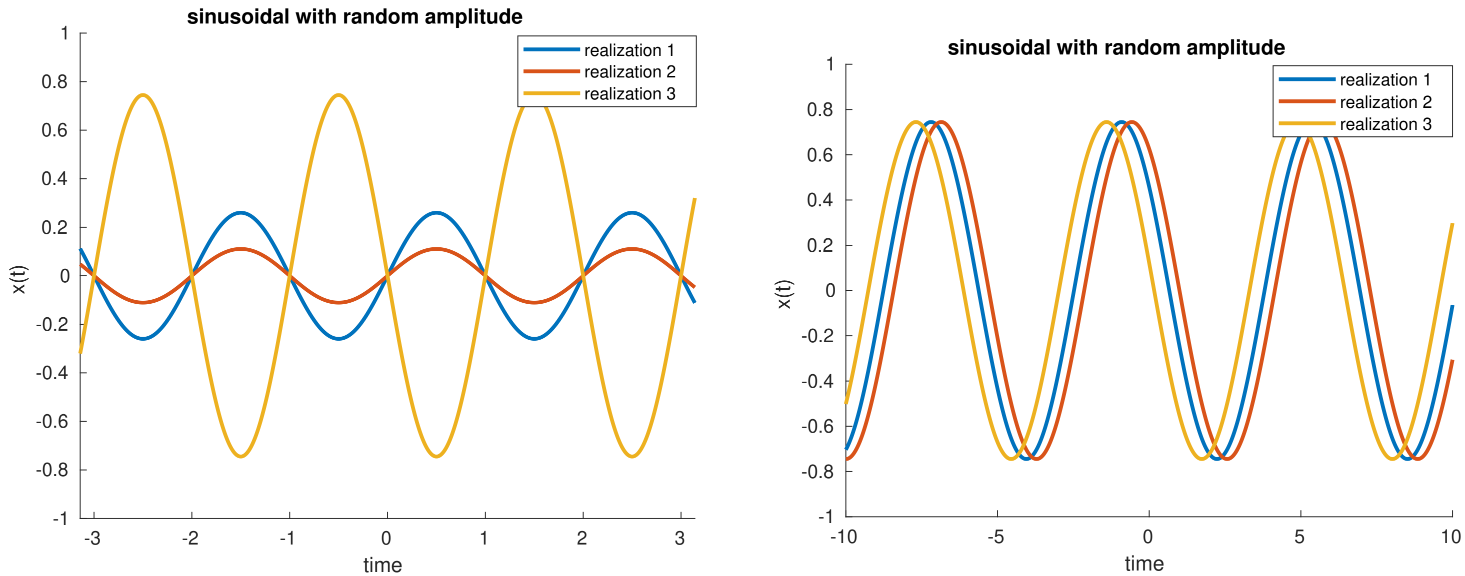
a family (or ensemble) of random variables indexed by t

$$\{X(t, \Theta), t \in I\}$$

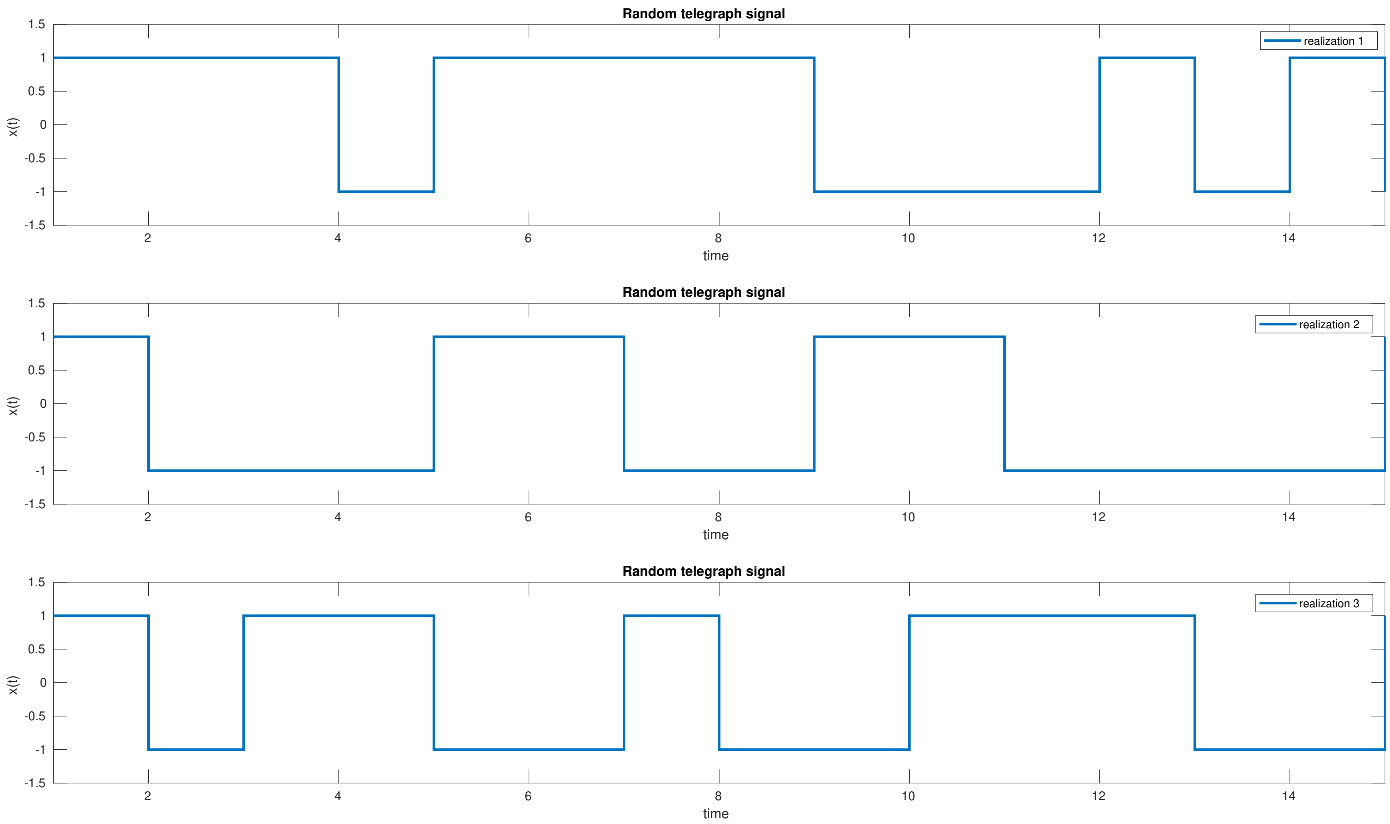
is called a **random (or stochastic) process**

$X(t, \Theta)$ when Θ is fixed, is called *a realization or sample path*

example: sinusoidal wave forms with random amplitude and phase



example: random telegraph signal



Specifying RPs

consider an RP $\{X(t,\Theta), t \in T\}$ when Θ is mapped from a sample space S

we often use the notation $X(t)$ to refer to an RP (just drop Θ)

- if T is a countable set then $X(t,\Theta)$ is called **discrete-time** RP
- if T is an uncountable set then $X(t,\Theta)$ is called **continuous-time** RP
- if S is a countable set then $X(t,\Theta)$ is called **discrete-valued** RP
- if S is an uncountable set then $X(t,\Theta)$ is called **continuous-valued** RP

another notation for discrete-time RP is $X[n]$ where n is the time index

From RV to RP

terms	RV	RP
cdf	$F_X(x)$	$F_{X(t)}(x)$
pdf (continuous-valued)	$f_X(x)$	$f_{X(t)}(x)$
pmf (discrete-valued)	$p(x)$	$p(x)$
mean	$m = \mathbf{E}[X]$	$m(t) = \mathbf{E}[X(t)]$
autocorrelation	$\mathbf{E}[X^2]$	$R(t_1, t_2) = \mathbf{E}[X(t_1)X(t_2)]$
variance	$\mathbf{var}[X]$	$\mathbf{var}[X(t)]$
autocovariance		$C(t_1, t_2) = \mathbf{cov}[X(t_1), X(t_2)]$
cross-correlation	$\mathbf{E}[XY]$	$R_{XY}(t_1, t_2) = \mathbf{E}[X(t_1)Y(t_2)]$
cross-covariance	$\mathbf{cov}(X, Y)$	$C_{XY}(t_1, t_2) = \mathbf{cov}[X(t_1), Y(t_2)]$

Distribution functions of RP (time sampled)

let sampling RP $X(t, \Theta)$ at times $t_1, t_2, \ldots, t_k$

$$X_1 = X(t_1, \Theta), \quad X_2 = X(t_2, \Theta), \ldots, X_k = X(t_k, \Theta)$$

this $(X_1, \ldots, X_k)$ is a *vector* RV

cdf of continuous-valued RV

$$F(x_1, x_2, \ldots, x_k) = P[X(t_1) \leq x_1, \ldots, X(t_k) \leq x_k]$$

pdf of continuous-valued RV

$$f(x_1, x_2, \ldots, x_k) dx_1 \cdots dx_k = \\ P[x_1 < X(t_1) < x_1 + dx_1, \ldots, x_k < X(t_k) < x_k + dx_k]$$

pmf of discrete-valued RV

$$p(x_1, x_2, \ldots, x_k) = P[X(t_1) = x_1, \ldots, X(t_k) = x_k]$$

- we have specified distribution functions from any time samples of RV
- the distribution is specified by the collection of kth-order joint cdf/pdf/pmf
- we have droped notation $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$ to simply $f(x_1, \ldots, x_k)$

Statistics

the **mean** function of an RP is defined by

$$m(t) = \mathbf{E}[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

the **variance** is defined by

$$\mathbf{var}[X(t)] = \mathbf{E}[(X(t) - m(t))^2] = \int_{-\infty}^{\infty} (x - m(t))^2 f_{X(t)}(x) dx$$

- both mean and variance functions are *deterministic* functions of time
- for discrete-time RV, another notation may be used: $m[n]$ where n is time index

the **autocorrelation** of $X(t)$ is the joint moment of RP at different times

$$R(t_1, t_2) = \mathbf{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} xy f_{X(t_1)X(t_2)}(x, y) dx dy$$

the **autocovariance** of $X(t)$ is the covariance of $X(t_1)$ and $X(t_2)$

$$C(t_1, t_2) = \mathbf{E}[(X(t_1) - m(t_1))(X(t_2) - m(t_2))]$$

relations:

- $C(t_1, t_2) = R(t_1, t_2) - m(t_1)m(t_2)$
- $\mathbf{var}[X(t)] = C(t, t)$

another notation for discrete-time RV: $R(m, n)$ or $C(m, n)$ where m, n are (integer) time indices

Joint distribution of RPs

let $X(t)$ and $Y(t)$ be two RPs

let $(t_1, \dots, t_k)$ and $(\tau_1, \dots, \tau_k)$ be time samples of $X(t)$ and $Y(t)$, resp.

we specify joint distribution of $X(t)$ and $Y(t)$ from *all possible time choices* of time samples of two RPs

$$
\begin{aligned}
f_{XY}(x_1, \dots, x_k, y_1, \dots, y_k) dx_1 \cdots dx_k dy_1 \cdots dy_k = \\
P[x_1 < X(t_1) \leq x_1 + dx_1, \dots, x_k < X(t_k) \leq x_k + dx_k, \\
y_1 < Y(\tau_1) \leq y_1 + dy_1, \dots, y_k < Y(\tau_k) \leq y_k + dy_k]
\end{aligned}
$$

note that time indices of $X(t)$ and $Y(t)$ need not be the same

Statistics of multiple RPs

the **cross-correlation** of $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(t_1, t_2) = \mathbf{E}[X(t_1)Y(t_2)]$$

(correlation of two RPs at different times)

the **cross-covariance** of $X(t)$ and $Y(t)$ is defined by

$$C_{XY}(t_1, t_2) = \mathbf{E}[(X(t_1) - m_X(t_1))(Y(t_2) - m_Y(t_2))]$$

relation: $C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2)$

more definitions:

two RPs $X(t)$ and $Y(t)$ are said to be

- **independent** if

  their joint cdf can be written as a product of two marginal cdf's

  mathematically,

$$F_{XY}(x_1, \ldots, x_k, y_1, \ldots, y_k) = F_X(x_1, \ldots, x_k) F_Y(y_1, \ldots, y_k)$$

- **uncorrelated** if

$$C_{XY}(t_1, t_2) = 0, \quad \text{for all } t_1 \text{ and } t_2$$

- **orthogonal** if

$$R_{XY}(t_1, t_2) = 0, \quad \text{for all } t_1 \text{ and } t_2$$

Stationary process

an RP is said to be **stationary** if the kth-order joint cdf's of

$$X(t_1), \ldots, X(t_k), \quad \text{and} \quad X(t_1 + \tau), \ldots, X(t_k + \tau)$$

are the *same*, for all time shifts τ and all k and all choices of $t_1, \ldots, t_k$

in other words, randomness of RP does not change with time

results: a stationary process has the following properties

- the mean is constant and independent of time: $m(t) = m$ for all t
- the variance is constant and independent of time

more results on stationary process:

- the first-order cdf is independent of time

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x), \quad \forall t, \tau$$

- the second-order cdf only depends on the time difference between samples

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2), \quad \forall t_1, t_2$$

- the autocovariance and autocorrelation can depend only on $t_2 - t_1$

$$R(t_1, t_2) = R(t_2 - t_1), \quad C(t_1, t_2) = C(t_2 - t_1), \quad \forall t_1, t_2$$

Wide-sense stationary process

if an RP $X(t)$ has the following two properties:

- the mean is constant: $m(t) = m$ for all t
- the autocovariance is a function of $t_2 - t_1$ only:

$$C(t_1, t_2) = C(t_1 - t_2), \quad \forall t_1, t_2$$

then $X(t)$ is said to be *wide-sense* stationary (WSS)

- all stationary RPs are wide-sense stationary (converse is not true)
- WSS is related to the concept of spectral density (later discussed)

Independent identically distributed processes

let $X[n]$ be a discrete-time RP and for any time instances $n_1, \dots, n_k$

$$X_1 = X[n_1], X_2 = X[n_2], \quad X_k = X[n_k]$$

definition: iid RP $X[n]$ consists of a sequence of independent, identically distributed (iid) random variables

$$X_1, X_2, \dots, X_k$$

with *common* cdf (in other words, same statistical properties)

this property is commonly assumed in applications for simplicity results: an iid process has the following properties

- the joint cdf of any time instances factors to the product of cdf's

$$F(x_1, \dots, x_k) = P[X_1 \leq x_1, \dots, X_k \leq x_k] = F(x_1)F(x_2)\cdots F(x_k)$$

- the mean is constant

$$m[n] = \mathbf{E}[X[n]] = m, \quad \forall n$$

- the autocovariance function is a delta function

$$C(n_1, n_2) = 0, \quad \text{for } n_1 \neq n_2, \quad C(n, n) = \sigma^2 \triangleq \mathbf{E}[(X[n] - m))^2]$$

- the autocorrelation function is given by

$$R(n_1, n_2) = C(n_1, n_2) + m^2$$

Independent and stationary increment property

let $X(t)$ be an RP and consider the interval $t_1 < t_2$

defitions:

- $X(t_2) - X(t_1)$ is called the **increment** of RP in the interval $t_1 < t < t_2$
- $X(t)$ is said to have **independent increments** if

  $$X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_k) - X(t_{k-1})$$

  are *independent* RV where $t_1 < t_2 < \cdots < t_k$ (non-overlapped times)
- $X(t)$ is said to have **stationary increments** if

  $$P[X(t_2) - X(t_1) = y] = P[X(t_2 - t_1) = y]$$

  the increments in intervals of the same length have the same distribution regardless of when the interval begins results:

- the joint pdf of $X(t_1), \ldots, X(t_k)$ is given by the product of pdf of $X(t_1)$ and the marginals of individual *increments*

we will see this result in the properties of a sum process

Jointly stationary process

$X(t)$ and $Y(t)$ are said to be jointly stationary if the joint cdf's of

$$X(t_1, \ldots, t_k) \text{ and } Y(\tau_1, \ldots, \tau_k)$$

do not depend on the time origin for all k and all choices of $(t_1, \ldots, t_k)$ and $(\tau_1, \ldots, \tau_k)$

Periodic and Cyclostationary processes

$X(t)$ is called **wide-sense periodic** if there exists $T > 0$,

- $m(t) = m(t+T)$ for all t (mean is periodic)
- $C(t_1, t_2) = C(t_1 + T, t_2) = C(t_1, t_2 + T) = C(t_1 + T, t_2 + T)$,
  for all t_1, t_2, (covariance is periodic in each of *two arguments*)

$X(t)$ is called **wide-sense cyclostationary** if there exists $T > 0$,

- $m(t) = m(t+T)$ for all t (mean is periodic)
- $C(t_1, t_2) = C(t_1 + T, t_2 + T)$ for all t_1, t_2
  (covariance is periodic in *both of two arguments*)

facts:

- sample functions of a wide-sense periodic RP are periodic with probability 1
$$X(t) = X(t+T), \quad \text{for all } t$$
except for a set of outcomes of probability zero

- sample functions of a wide-sense cyclostationary RP need NOT be periodic

examples:

- sinusoidal signal with random amplitude (page 10-4) is wide-sense cyclostationary and sample functions are periodic

- PAM signal (page 10-19) is wide-sense cyclostationary but sample functions are not periodic

Stochastic periodicity

definition: a continuous-time RP $X(t)$ is **mean-square periodic** with period T, if

$$\mathbf{E}[(X(t+T) - X(t))^2] = 0$$

let $X(t)$ be a wide-sense stationary RP

$X(t)$ is mean-square periodic if and only if

$$R(\tau) = R(\tau + T), \quad \text{for all } \tau$$

$i.e.$, its autocorrelation function is periodic with period T

Ergodic random process

the **time average** of a realization of a WSS RP is defined by

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

the time-average autocorrelation function is defined by

$$\langle x(t)x(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau) dt$$

- if the time average is equal to the ensemble average, we say the RP is **ergodic in mean**

- if the time-average autocorrelation is equal to ensemble autocorrelation then the RP is **ergodic in the autocorrelation**

definition: a WSS RP is **ergodic** if ensemble averages can be calculated using time averages of any realization of the process

- ergodic in mean: $\langle x(t) \rangle = \mathbf{E}[X(t)]$
- ergodic in autocorrelation: $\langle x(t)x(t+\tau) \rangle = \mathbf{E}[X(t)X(t+\tau)]$

calculus of random process (derivative, integrals) is discussed in mean-square sense

see Leon-Garcia, Section 9.7.2-9.7.3