10. Examples of random processes

- sinusoidal signals
- random telegraph signals
- signal plus noise
- ARMA time series

Sinusoidal signals

consider a signal of the form

$$X(t) = A\sin(\omega t + \phi)$$

in each of following settings:

- ω (frequencey) is random
- A (amplitude) is random
- ϕ (phase shift) is random

questions involving this example

- find pdf
- find mean, variance, and correlation function

random frequency:

- sample functions are sinusoidal signals at different frequencies

random amplitude: $A \in \mathcal{U}[-1, 1]$ while $\omega = \pi$ and $\phi = 0$

$$X(t) = A\sin(\pi t)$$

(continuous-valued RP and sample function is periodic)

- find pdf of $X(t)$
  - when t is integer, we see $X(t) = 0$ for all A

$$P(X(t) = 0) = 1, \quad P(X(t) = \text{other values}) = 0$$

  - when t is not integer, $X(t)$ is just a scaled uniform RV

$$X(t) \in \mathcal{U}[-\sin(\pi t), \sin(\pi t)]$$

- find mean of $X(t)$

$$m(t) = \mathbf{E}[X(t)] = \mathbf{E}[A]\sin(\pi t)$$

(could have zero mean if $\sin(\pi t) = 0$)

- find correlation function

$$\begin{aligned} R(t_1, t_2) &= \mathbf{E}[A\sin(\pi t_1)A\sin(\pi t_2)] \\ &= \mathbf{E}[A^2]\sin(\pi t_1)\sin(\pi t_2) \end{aligned}$$

- find covariance function

$$\begin{aligned} C(t_1, t_2) &= R(t_1, t_2) - m(t_1)m(t_2) \\ &= \mathbf{E}[A^2]\sin(\pi t_1)\sin(\pi t_2) - (\mathbf{E}[A])^2\sin(\pi t_1)\sin(\pi t_2) \\ &= \mathbf{var}[A]\sin(\pi t_1)\sin(\pi t_2) \end{aligned}$$

- $X(t)$ is wide-sense cyclostationary, $i.e.$, $m(t) = m(t+T)$ and $C(t_1, t_2) = C(t_1 + T, t_2 + T)$ for some T

random phase shift: $A = 1, \omega = 1$ and $\phi \sim \mathcal{U}[-\pi, \pi]$

$$X(t) = \sin(t + \phi)$$

(continuous-valued RP)

- find pdf of $X(t)$: view $x = \sin(t + \phi)$ as a transformation of ϕ

$$x = \sin(t + \phi) \Leftrightarrow \phi_1 = \sin^{-1}(x) - t, \phi_2 = \pi - \sin^{-1}(x) - t$$

the pdf of $X(t)$ can be found from the formula

$$\begin{aligned} f_{X(t)}(x) &= \sum_k f(\phi_k) \left| \frac{d\phi}{dx} \right|_{\phi = \phi_k} \\ &= \frac{1}{\pi\sqrt{1 - x^2}}, \quad -1 \le x \le 1 \end{aligned}$$

(pdf of $X(t)$ does not depend on t; hence, $X(t)$ is strict-sense stationary)

- find the mean function

$$\mathbf{E}[X(t)] = \mathbf{E}[\sin(t+\phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(t+\phi)d\phi = 0$$

- find the covariance function

$$\begin{aligned} C(t_1,t_2) = R(t_1,t_2) &= \mathbf{E}[\sin(t_1+\phi)\sin(t_2+\phi)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos(t_1-t_2) - \cos(t_1+t_2+2\phi)]d\phi \\ &= (1/2)\cos(t_1-t_2) \end{aligned}$$

  (depend only on $t_1 - t_2$)

- $X(t)$ is wide-sense stationary (also conclude from the fact that $X(t)$ is stationary)

Random Telegraph signal

a signal $X(t)$ takes values in $\{1, -1\}$ randomly in the following setting:

- $X(0) = 1$ or $X(0) = -1$ with equal probability of $1/2$
- $X(t)$ changes the sign with each occurrence follows a Poisson process of rate α

questions involving this example:

- obviously, $X(t)$ is a discrete-valued RP, so let's find its pmf
- find the covariance function
- examine its stationary property

pmf of random telegraph signals: based on the fact that

$X(t)$ has the *same* sign as $X(0)$ $\iff$ number of sign changes in $[0,t]$ is *even*

$X(t)$ and $X(0)$ *differ* in sign $\iff$ number of sign changes in $[0,t]$ is *odd*

$$P(X(t)=1) = \underbrace{P(X(t)=1|X(0)=1)}_{\text{no. of sign change is even}} P(X(0)=1) + \underbrace{P(X(t)=1|X(0)=-1)}_{\text{no. of sign change is odd}} P(X(0)=-1)$$

let $N(t)$ be the number of sign changes in $[0,t]$ (which is Poisson)

$$P(N(t) = \text{ even integer}) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k}}{(2k)!} e^{-\alpha t} = (1/2)(1+e^{-2\alpha t})$$

$$P(N(t) = \text{ odd integer}) = \sum_{k=0}^{\infty} \frac{(\alpha t)^{2k+1}}{(2k+1)!} e^{-\alpha t} = (1/2)(1-e^{-2\alpha t})$$

pmf of $X(t)$ is then obtained as

$$\begin{aligned} P(X(t) = 1) &= (1/2)(1 + e^{-2\alpha t})(1/2) + (1/2)(1 - e^{-2\alpha t})(1/2) \\ &= 1/2 \\ P(X(t) = -1) &= 1 - P(X(t) = 1) = 1/2 \end{aligned}$$

- pmf of $X(t)$ does not depend on t
- $X(t)$ takes values in $\{-1, 1\}$ with equal probabilities

if $X(0) = 1$ with probability $p \neq 1/2$ then how the pmf of $X(t)$ would change ?

mean and variance:

- mean function:

$$\mathbf{E}[X(t)] = \sum_k x_k P(X(t) = x_k) = 1 \cdot (1/2) + (-1) \cdot (1/2) = 0$$

- variance:

$$\begin{aligned} \mathbf{var}[X(t)] = \mathbf{E}[X(t)^2] - (\mathbf{E}[X(t)])^2 &= \sum_k x_k^2 P(X(t) = x_k) \\ &= (1)^2 \cdot (1/2) + (-1)^2 \cdot (1/2) = 1 \end{aligned}$$

both mean and variance functions do not depend on time

covariance function: since mean is zero and by definition

$$\begin{aligned} C(t_1, t_2) &= \mathbf{E}[X(t_1)X(t_2)] \\ &= (1)^2 P[X(t_1) = 1, X(t_2) = 1] + (-1)^2 P[X(t_1) = -1, X(t_2) = -1] \\ &+ (1)(-1)P[X(t_1) = 1, X(t_2) = -1] + (-1)(1)P[X(t_1) = -1, X(t_2) = 1] \end{aligned}$$

from above, we need to find joint pmf obtained via conditional pmf

$$P(X(t_1) = x_1, X(t_2) = x_2) = \underbrace{P(X(t_2) = x_2 \mid X(t_1) = x_1)}_{\text{depend on sign change}} \underbrace{P(X(t_1) = x_1)}_{\text{known}}$$

- $X(t_1)$ and $X(t_2)$ have the same sign

  $$P(X(t_2) = x_1 \mid X(t_1) = x_1) = P(N(t_2 - t_1) = \text{ even}) = (1/2)(1 + e^{-2\alpha(t_2 - t_1)})$$

- $X(t_1)$ and $X(t_2)$ have different signs

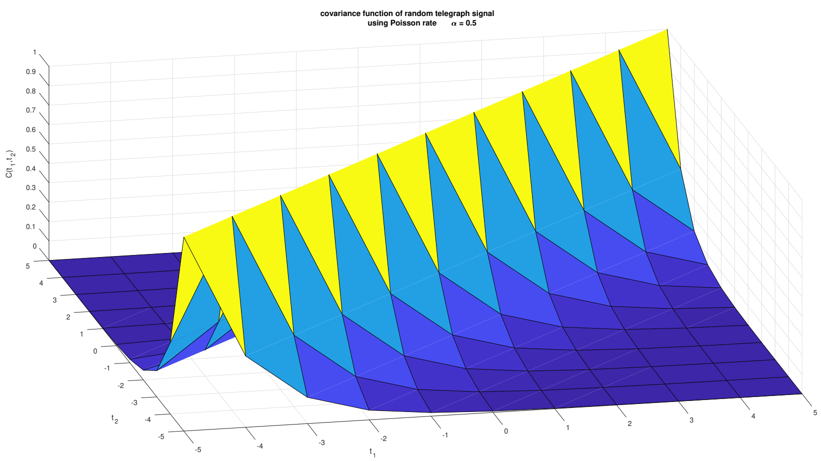
  $$P(X(t_2) = -x_1 \mid X(t_1) = x_1) = P(N(t_2 - t_1) = \text{ odd}) = (1/2)(1 - e^{-2\alpha(t_2 - t_1)})$$

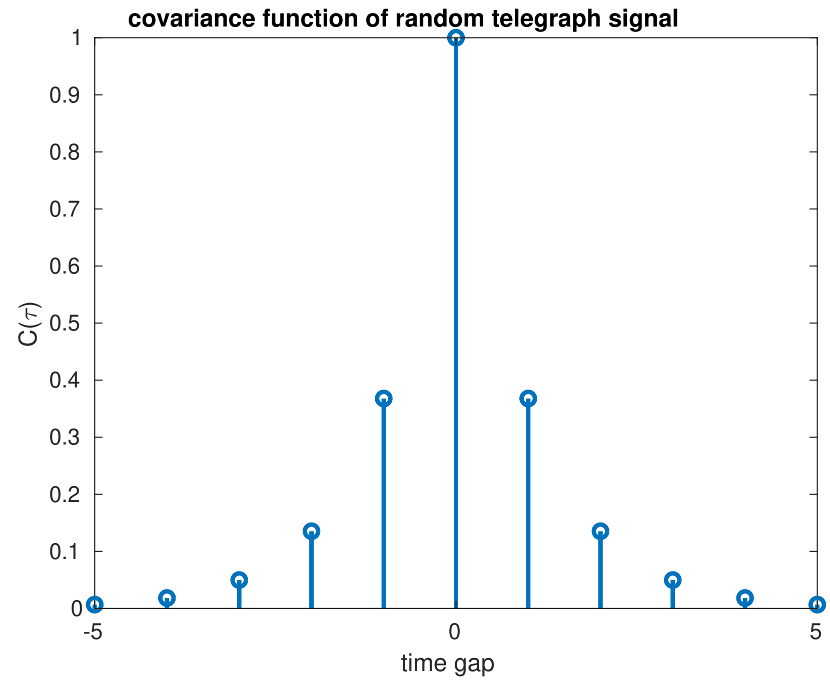
the covariance is obtained by

$$
\begin{aligned}
C(t_1, t_2) &= P(X(t_1) = X(t_2)) + P(X(t_1) \neq X(t_2)) \\
&= 2 \cdot (1/2)(1 + e^{-2\alpha(t_2 - t_1)})(1/2) - 2 \cdot (1/2)(1 - e^{-2\alpha(t_2 - t_1)}) \cdot (1/2) \\
&= e^{-2\alpha|t_2 - t_1|}
\end{aligned}
$$

- it depends only on the time gap $t_2 - t_1$, denoted as $\tau = t_2 - t_1$
- we can rewrite $C(\tau) = e^{-2\alpha|\tau|}$
- as $\tau \to \infty$, values of $X(t)$ at different times are less correlated
- $X(t)$ (based on the given setting) is wide-sense stationary

covariance function of random telegraph signal: set $\alpha = 0.5$





- left: $C(t_1, t_2) = e^{-2\alpha|t_2 - t_1|}$ as a function of (t_1, t_2)
- right: $C(t_1, t_2) = C(\tau)$ as a function of τ only

revisit telegraph signal: when $X(0) = 1$ with probability $p \neq 1/2$

- how would pmf of $X(t)$ change ?
- examine stationary property under this setting

pmf of $X(t)$

$$
\begin{aligned}
P(X(t) = 1) &= (1/2)(1 + e^{-2\alpha t})(p) + (1/2)(1 - e^{-2\alpha t})(1 - p) \\
&= 1/2 + e^{-2\alpha t}(p - 1/2) \\
P(X(t) = -1) &= 1 - P(X(t) = 1) \\
&= 1/2 - e^{-2\alpha t}(p - 1/2)
\end{aligned}
$$

- when $p \neq 1/2$, pmf of $X(t)$ varies over time but pmf converges to uniform as $t \to \infty$, regardless of the value of p
- if $p = 1$ ($X(0)$ is deterministic) then pmf still varies over time:

$$P(X(t) = 1) = (1/2)(1 + e^{-2\alpha t}), \quad P(X(t) = -1) = (1/2)(1 - e^{-2\alpha t})$$

stationary property: $X(t)$ is stationary if

$$P(X(t_1) = x_1, \ldots, X(t_k) = x_k) = P(X(t_1 + \tau) = x_k, \ldots, X(t_k + \tau) = x_k)$$

for any $t_1 < t_2 < \cdots < t_k$ and any τ

examine by characterizing pmf as product of conditional pmf's

$$p(x_1, \ldots, x_k) = p(x_k | x_{k-1}, \ldots, x_1) p(x_{k-1} | x_{k-2}, \ldots, x_1) \cdots p(x_2 | x_1) p(x_1)$$

which reduces to

$$P(X(t_1) = x_1, \ldots, X(t_k) = x_k) =$$
$$P(X(t_k) = x_k | X(t_{k-1}) = x_{k-1}) \cdots P(X(t_2) = x_2 | X(t_1) = x_1) P(X(t_1) = x_1)$$

using *independent increments property* of Poisson process because

- for example, if $X(t_k)$ don't change sign

$$P(X(t_k) = x_k | X(t_{k-1}) = x_{k-1}) = P(N(t_k - t_{k-1}) = \text{ even})$$

  if $X(t_{k-1})$ is given, values of $X(t_k)$ are determined solely by $N(t)$ in intervals (t_j, t_{j-1}) which is independent of the previous intervals

- only knowing x_{k-1} is enough to know conditional pmf:

$$P(x_k | x_{k-1}, x_{k-2}, \ldots, x_1) = P(x_k | x_{k-1})$$

then, we can find each of the conditional pmf's

$$P(X(t_k) = x_k | X(t_{k-1}) = x_{k-1}) = \begin{cases} (1/2)(1 + e^{-2\alpha(t_k - t_{k-1})}), \text{for } x_k = x_{k-1} \\ (1/2)(1 - e^{-2\alpha(t_k - t_{k-1})}), \text{for } x_k = -x_{k-1} \end{cases}$$

with the same reasoning, we can write the joint pmf (with time shift) as

$$
\begin{aligned}
P(X(t_1+\tau) = x_1, \ldots, X(t_k+\tau) = x_k) = \\
P(X(t_k+\tau) = x_k | X(t_{k-1}+\tau) = x_{k-1}) \cdots \\
P(X(t_2+\tau) = x_2 | X(t_1+\tau) = x_1) P(X(t_1+\tau) = x_1)
\end{aligned}
$$

where these are equal

$$
P(X(t_k) = x_k | X(t_{k-1}) = x_{k-1}) = P(X(t_k+\tau) = x_k | X(t_{k-1}+\tau) = x_{k-1})
$$

because it depends only on the time gap (from page 10-17)

as a result, to examine stationary property, we only need to compare

$$
P(X(t_1) = x_1) \quad \text{VS} \quad P(X(t_1+\tau) = x_1)
$$

which only equal in **steady-state sense** (as $t_1 \to \infty$) from page 10-15

Pulse amplitude modulation (PAM)

setting: to send a sequence of binary data, transmit 1 or -1 for T seconds

$$X(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT)$$

where A_k is random amplitude (± 1) and $p(t)$ is a pulse of width T

- $m(t) = 0$ since $\mathbf{E}[A_n] = 0$
- $C(t_1, t_2)$ is given by

$$C(t_1,t_2) = \begin{cases} \mathbf{E}[X(t_1)^2] = 1, & \text{if } nT \leq t_1, t_2 < (n+1)T \\ \mathbf{E}[X(t_1)]\mathbf{E}[X(t_2)] = 0, & \text{otherwise} \end{cases}$$

- $X(t)$ is wide-sense cyclostationary but clearly sample function of $X(t)$ is not periodic

Signal with additive noise

most applications encounter a random process of the form

$$Y(t) = X(t) + W(t)$$

- $X(t)$ is transmitted signal (could be deterministic or random) but unknown
- $Y(t)$ is the measurement (observable to users)
- $W(t)$ is noise that corrupts the transmitted signal

common questions regarding this model:

- if only $Y(t)$ is measurable can we reconstruct/estimate what $X(t)$ is ?
- if we can, what kind of statistical information about $W(t)$ do we need ?
- if $X(t)$ is deterministic, how much W affect to Y in terms of fluctuation?

simple setting: let us make X and W independent

cross-covariance: let $\tilde{X}$ and $\tilde{W}$ be the mean removed versions

$$
\begin{aligned}
C_{xy}(t_1, t_2) &= \mathbf{E}[(X(t_1) - m_x(t_1))(Y(t_2) - m_y(t_2))] \\
&= \mathbf{E}[(X(t_1) - m_x(t_1))((X(t_2) + W(t_2) - m_x(t_2) - m_w(t_2))] \\
&= \mathbf{E}[\tilde{X}(t_1)(\tilde{X}(t_2) + \tilde{W}(t_2))] \\
&= C_x(t_1, t_2) + 0
\end{aligned}
$$

cross-covariance can never be zero as Y is a function of X

autocovariance:

$$
\begin{aligned}
C_y(t_1, t_2) &= \mathbf{E}[(Y(t_1) - m_y(t_1))(Y(t_2) - m_y(t_2))] \\
&= \mathbf{E}[(\tilde{X}(t_1) + \tilde{W}(t_1))(\tilde{X}(t_2) + \tilde{W}(t_2))] \\
&= C_x(t_1, t_2) + C_w(t_1, t_2) + 0
\end{aligned}
$$

the variance in Y is always higher than X; the increase is from the noise