4. Pairs of Random Variables

- probabilities
- conditional probability and expectation
- independence
- joint characteristic function
- functions of random variables

Definition

let ζ be an outcome in the sample space S

a pair of RVs $Z(\zeta)$ is a function that maps ζ to a pair of real numbers

$$Z(\zeta) = (X(\zeta), Y(\zeta))$$

example: a web page provides the user with a choice either to watch an ad or move directly to the requested page

let ζ be the patterns of user arrivals to a webpage

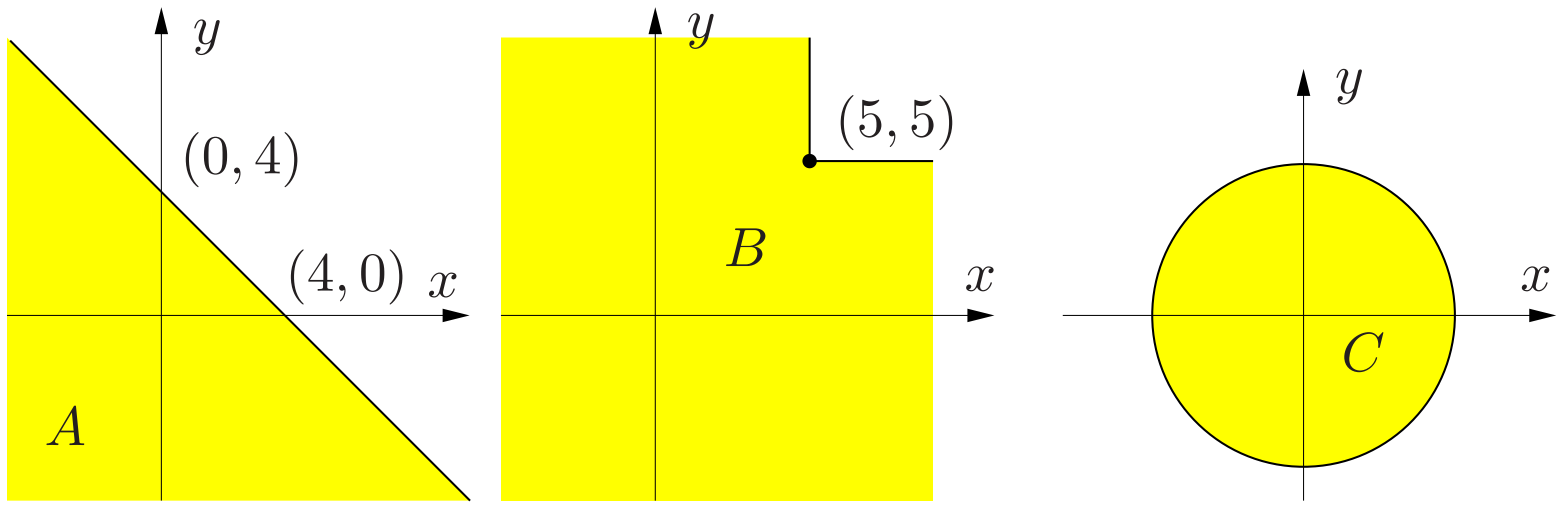
- $N_1(\zeta)$ be the number of times the webpage is directly requested
- $N_2(\zeta)$ be the number of that the ads is chosen

$(N_1(\zeta), N_2(\zeta))$ assigns a pair of nonnegative integers to each outcome ζ

Events of interest

events involving a pair of RVs (X, Y) can be represented by regions

Example:



$$A = \{X + Y \le 4\}, \quad B = \{\min(X, Y) \le 5\}, \quad C = \{X^2 + Y^2 \le 25\}$$

- A: total revenue from two sources is less than 4
- C: total noise power is less than r^2

Events and Probabilities

we consider the events that that the product form:

$$C = \{X \text{ in } A\} \cap \{Y \text{ in } B\}$$

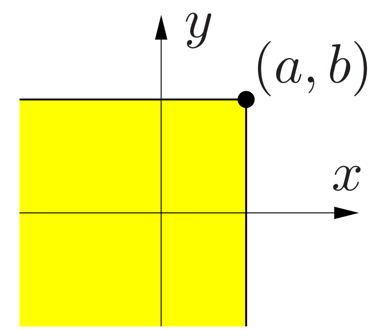
the probability of product-form events is

$$\begin{aligned} P(C) &= P(\{X \text{ in } A\} \cap \{Y \text{ in } B\}) \\ &= P(X \text{ in } A,\ Y \text{ in } B) \end{aligned}$$

Probability for pairs of random variables

Joint cumulative distribution function

$$F_{XY}(a,b) = P(X \le a, Y \le b)$$



Properties:

- a joint CDF is a nondecreasing function of x and y:

$$F_{XY}(x_1,y_1) \le F_{XY}(x_2,y_2), \quad \text{if } x_1 \le x_2 \text{ and } y_1 \le y_2$$

- $F_{XY}(x_1, -\infty) = 0$, $F_{XY}(-\infty, y_1) = 0$, $F_{XY}(\infty, \infty) = 1$
- $P(x_1 < X \le x_2, y_1 < Y \le y_2)$

$$= F_{XY}(x_2,y_2) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2) + F_{XY}(x_1,y_1)$$

Joint PMF for discrete RVs

$$p_{XY}(x,y) = P(X = x, Y = y), \quad (x,y) \in S$$

Joint PDF for continuous RVs

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Marginal PMF

$$p_X(x) = \sum_{y \in S} p_{XY}(x,y), \quad p_Y(y) = \sum_{x \in S} p_{XY}(x,y)$$

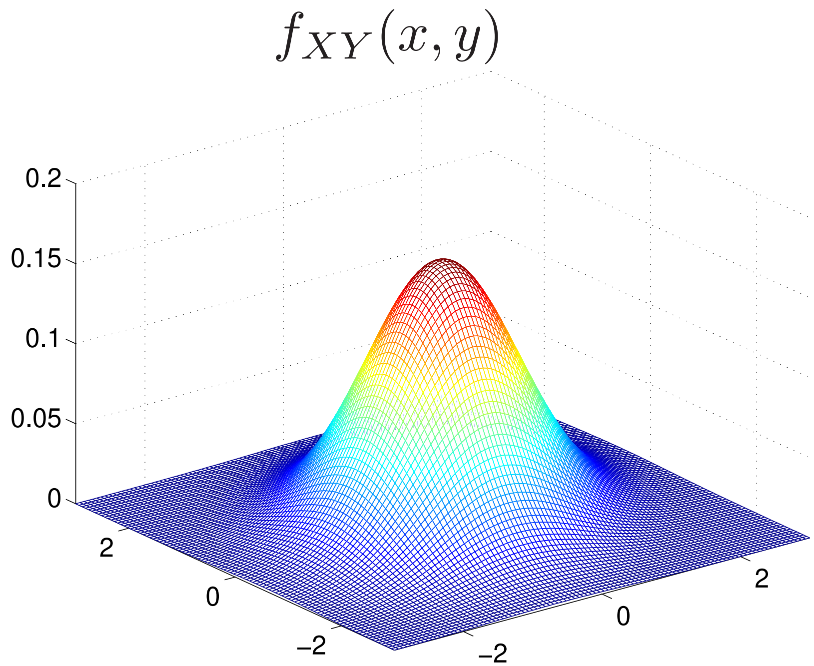
Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,z)dz, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(z,y)dz$$

Example 1: Jointly Gaussian Random Variables

if X, Y are jointly Gaussian, a joint pdf of X and Y can be given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \text{ exp } -\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}, \quad -\infty < x, y < \infty$$



with $|\rho| < 1$

find the marginal PDF's

the marginal pdf of X is found by integrating $f_{XY}(x, y)$ over y:

$$
\begin{aligned}
f_X(x) &= \frac{e^{-x^2/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-(y^2-2\rho xy)/2(1-\rho^2)} dy \\
&= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-(y-\rho x)^2/2(1-\rho^2)}}{\sqrt{2\pi(1-\rho^2)}} dy \\
&= \frac{e^{-x^2/2}}{\sqrt{2\pi}}
\end{aligned}
$$

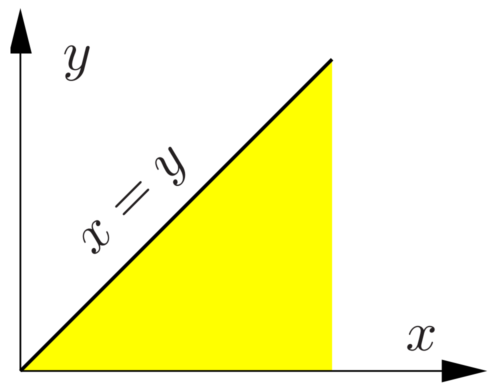
- the second step follows from completing the square in $(y-\rho x)^2$
- the last integral equals 1 since its integrand is a Gaussian pdf with mean ρx and variance $1-\rho^2$
- the marginal pdf of X is also a Gaussian with mean 0 and variance 1
- from the symmetry of $f_{XY}(x, y)$ in x and y, the marginal pdf of Y is also the same as X

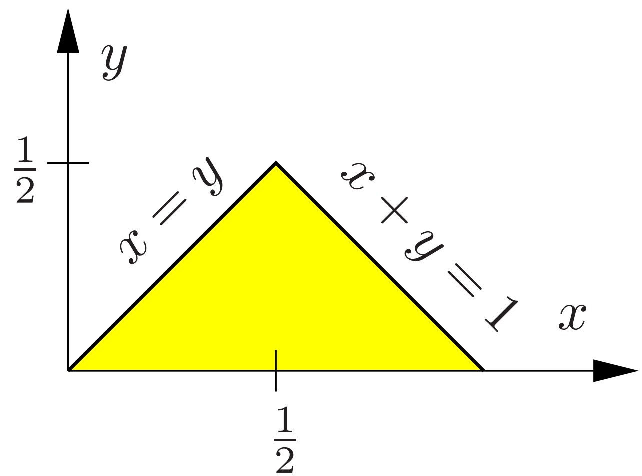
Example 2

consider X and Y with a joint PDF

$$f_{XY}(x,y) = ce^{-x}e^{-y}, \quad 0 \leq y \leq x < \infty$$

find the constant c, the marginal PDFs and $P(X+Y \leq 1)$





the constant c is found from the normalization condition:

$$1 = \int_0^\infty \int_0^x ce^{-x}e^{-y}dydx \Longrightarrow c = 2$$

the marginal PDFs are obtained by

$$f_X(x) = \int_0^\infty f_{XY}(x,y)dy = \int_0^x 2e^{-x}e^{-y}dy, \quad 0 \le x < \infty$$

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)dx = \int_y^\infty 2e^{-x}e^{-y}dx = 2e^{-y}, \quad 0 \le y < \infty$$

$P(X+Y \le 1)$ can be found by taking the intersection of the region where the joint PDF is nonzero and the event $\{X+Y \le 1\}$

$$\begin{aligned} P(X+Y \le 1) &= \int_0^{1/2}\int_y^{1-y} 2e^{-x}e^{-y}dxdy = \int_0^{1/2} 2e^{-y}[e^{-y} - e^{-(1-y)}]dy \\ &= 1 - 2e^{-1} \end{aligned}$$

Conditional Probability

Discrete RVs

the *conditional PMF of* Y *given* $X = x$ is defined by

$$
\begin{aligned}
p_{Y|X}(y|x) &= P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)} \\
&= \frac{p_{XY}(x, y)}{p_X(x)}
\end{aligned}
$$

Continuous RVs

the *conditional PDF of* Y *given* $X = x$ is defined by

$$
f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}
$$

Example: Number of defects in a region

- let X be the total number of defects on a chip

$$X \sim \text{Poisson}(\alpha)$$

- let Y be the number of defects falling in region R
- if $X = n$ (given), then Y is binomial with (n, p)

$$p_{Y|X}(k|n) = \begin{cases} 0, & k > n \\ \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}, & 0 \leq k \leq n \end{cases}$$

- we can show that

$$Y \sim \text{Poisson}(\alpha p)$$

$$
\begin{aligned}
P(Y=k) &= \sum_{n=0}^{\infty} P(Y=k|X=n)P(X=n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\alpha^n e^{-\alpha}}{n!} \\
&= \frac{(\alpha p)^k e^{-\alpha}}{k!} \sum_{n=k}^{\infty} \frac{[(1-p)\alpha]^{n-k}}{(n-k)!} \\
&= \frac{(\alpha p)^k e^{-\alpha} e^{(1-p)\alpha}}{k!} = \frac{(\alpha p)^k}{k!} e^{-\alpha p}
\end{aligned}
$$

Example: Customers arrive at a service station

- let N be # of customers arriving at a station during time t

$$N \sim \text{Poisson}(\beta t)$$

- let T be the service time for each customer

$$T \sim \text{exponential}(\alpha)$$

- we can show that # of customers that arrive during the service time is a geometric RV with probability of success $\alpha/(\alpha + \beta)$

$$
\begin{aligned}
P(N=k) &= \int_0^\infty P(N=k|T=t) f_T(t)\,dt \\
&= \int_0^\infty \left(\frac{(\beta t)^k}{k!} e^{-\beta t} \right) \alpha e^{-\alpha t} dt \\
&= \frac{\alpha \beta^k}{k!} \int_0^\infty t^k e^{-(\alpha+\beta)t} dt
\end{aligned}
$$

let $r = (\alpha+\beta)t$, then

$$
\begin{aligned}
P(N=k) &= \frac{\alpha\beta^k}{k!(\alpha+\beta)^{k+1}} \int_0^\infty r^k e^{-r} dr \\
&= \left(\frac{\alpha}{\alpha+\beta} \right) \left(\frac{\beta}{\alpha+\beta} \right)^k
\end{aligned}
$$

(the last integral is a gamma function and is equal to $k!$)

Conditional Expectation

the conditional expectation of Y given $X = x$ is defined by

Continuous RVs

$$\mathbf{E}[Y|X] = \int_{-\infty}^{\infty} y \; f_{Y|X}(y|x) dy$$

Discrete RVs

$$\mathbf{E}[Y|X] = \sum_{y} y \; p_{Y|X}(y|x)$$

- $\mathbf{E}[Y|X]$ is the center of mass associated with the conditional pdf or pmf
- $\mathbf{E}[Y|X]$ can be viewed as a function of random variable X
- $\mathbf{E}[\mathbf{E}[Y|X]] = \mathbf{E}[Y]$

in fact, we can show that

$$\mathbf{E}[h(Y)] = \mathbf{E}[\mathbf{E}[h(Y)|X]]$$

for any function $h(\cdot)$ that $\mathbf{E}[|h(Y)|] < \infty$

proof.

$$\begin{aligned}
\mathbf{E}[\mathbf{E}[h(Y)|X]] &= \int_{-\infty}^{\infty} \mathbf{E}[h(Y)|x] f_X(x) dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(y) f_{Y|X}(y|x) dy \; f_X(x) dx \\
&= \int_{-\infty}^{\infty} h(y) \int_{-\infty}^{\infty} f_{XY}(x,y) \; dx dy \\
&= \int_{-\infty}^{\infty} h(y) f_Y(y) \; dy \\
&= \mathbf{E}[h(Y)]
\end{aligned}$$

Example: Average defects of a chip in a region

From the example on page 4-12,

$$
\begin{aligned}
E[Y] &= \mathbf{E}[\mathbf{E}[Y|X]] \\
&= \sum_{n=0}^{\infty} np\, P(X=n) \\
&= p \sum_{n=0}^{\infty} nP(X=n) \\
&= p\mathbf{E}[X] = \alpha p
\end{aligned}
$$

Example: Average arrivals in a service time

from the example on page 4-14,

$$
\begin{aligned}
\mathbf{E}[N] &= \mathbf{E}[\mathbf{E}[N|T]] \\
&= \int_0^\infty \mathbf{E}[N|T=t] f_T(t)dt \\
&= \int_0^\infty \beta t f_T(t)dt \\
&= \beta \mathbf{E}[T] \\
&= \frac{\beta}{\alpha}
\end{aligned}
$$

Example: Variance of arrivals in a service time

same example as in page 4-12

N is Poisson RV with parameter βt when $T = t$ is given, so

$$\mathbf{E}[N|T = t] = \beta t, \quad \mathbf{E}[N^2|T = t] = (\beta t) + (\beta t)^2$$

the second moment of N can be calculated by

$$\begin{aligned} \mathbf{E}[N^2] &= \mathbf{E}[\mathbf{E}[N^2|T]] \\ &= \int_0^\infty \mathbf{E}[N^2|T = t]\, f_T(t)\, dt \\ &= \int_0^\infty (\beta t + \beta^2 t^2)\, f_T(t)\, dt \\ &= \beta \mathbf{E}[T] + \beta^2 \mathbf{E}[T^2] \end{aligned}$$

therefore,

$$\begin{aligned}
\mathbf{var}(N) &= \mathbf{E}[N^2] - (\mathbf{E}[N])^2 \\
&= \beta^2\mathbf{E}[T^2] + \beta\mathbf{E}[T] - \beta^2(\mathbf{E}[T])^2 \\
&= \beta^2\,\mathbf{var}(T) + \beta\mathbf{E}[T]
\end{aligned}$$

- if T is not random ($\mathbf{E}[T]$ is constant and $\mathbf{var}(T) = 0$), the mean and variance of N are those of a Poisson RV with parameter $\beta\mathbf{E}[T]$
- when T is random, the mean of N remains the same, but $\mathbf{var}(N)$ increases by the term $\beta^2\,\mathbf{var}(T)$
- bote that the above result holds for *any* distribution $f_T(t)$
- if T is exponential with parameter α, then $\mathbf{E}[T] = 1/\alpha$ and $\mathbf{var}(T) = 1/\alpha^2$, so

$$\mathbf{E}[N] = \frac{\beta}{\alpha}, \quad \mathbf{var}(N) = \frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha}$$

Independence of two random variables

X and Y are independent if and only if

$$F_{XY}(x,y) = F_X(x)F_Y(y), \quad \forall x, y$$

this is equivalent to

Discrete Random Variables

$$\begin{aligned} p_{XY}(x,y) &= p_X(x)p_Y(y) \\ p_{Y|X}(y|x) &= p_Y(y) \end{aligned}$$

Continuous Random Variables

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$f_{Y|X}(y|x) = f_{Y|X}(y)$$

If X and Y are independent, so are any pair of functions $g(X)$ and $h(Y)$

Example

let X and Y be Gaussian RVs with zero mean and unit variance

the product of the marginal pdf's of X and Y is

$$f_X(x)f_Y(y) = \frac{1}{2\pi}\,\text{exp}\,-\frac{(x^2+y^2)}{2}, \quad -\infty < x, y < \infty$$

from the example on page 4-7, the joint pdf of X and Y is

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\,\text{exp}\,-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}, \quad -\infty < x, y < \infty$$

therefore the jointly Guassian X and Y are independent if and only if

$$\rho = 0$$

ρ is called **correlation coefficient** between X and Y

Expected Values and Covariance

the expected value of $Z = g(X,Y)$ is defined as

$$\begin{aligned}
\mathbf{E}[Z] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy & \quad X,Y \text{ continuous} \\
\mathbf{E}[Z] &= \sum_x \sum_y g(x,y)\, p_{XY}(x,y) & \quad X,Y \text{ discrete}
\end{aligned}$$

- $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$
- $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ if X and Y are independent

Covariance of X and Y

$$\mathbf{cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$$

- $\mathbf{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$
- $\mathbf{cov}(X,Y) = 0$ if X and Y are independent (the converse is NOT true)

Correlation Coefficient

denote

$$\sigma_X = \sqrt{\mathbf{var}(X)}, \quad \sigma_Y = \sqrt{\mathbf{var}(Y)}$$

the standard deviations of X and Y

the **correlation coefficient** of X and Y is defined by

$$\rho_{XY} = \frac{\mathbf{cov}(X, Y)}{\sigma_X \sigma_Y}$$

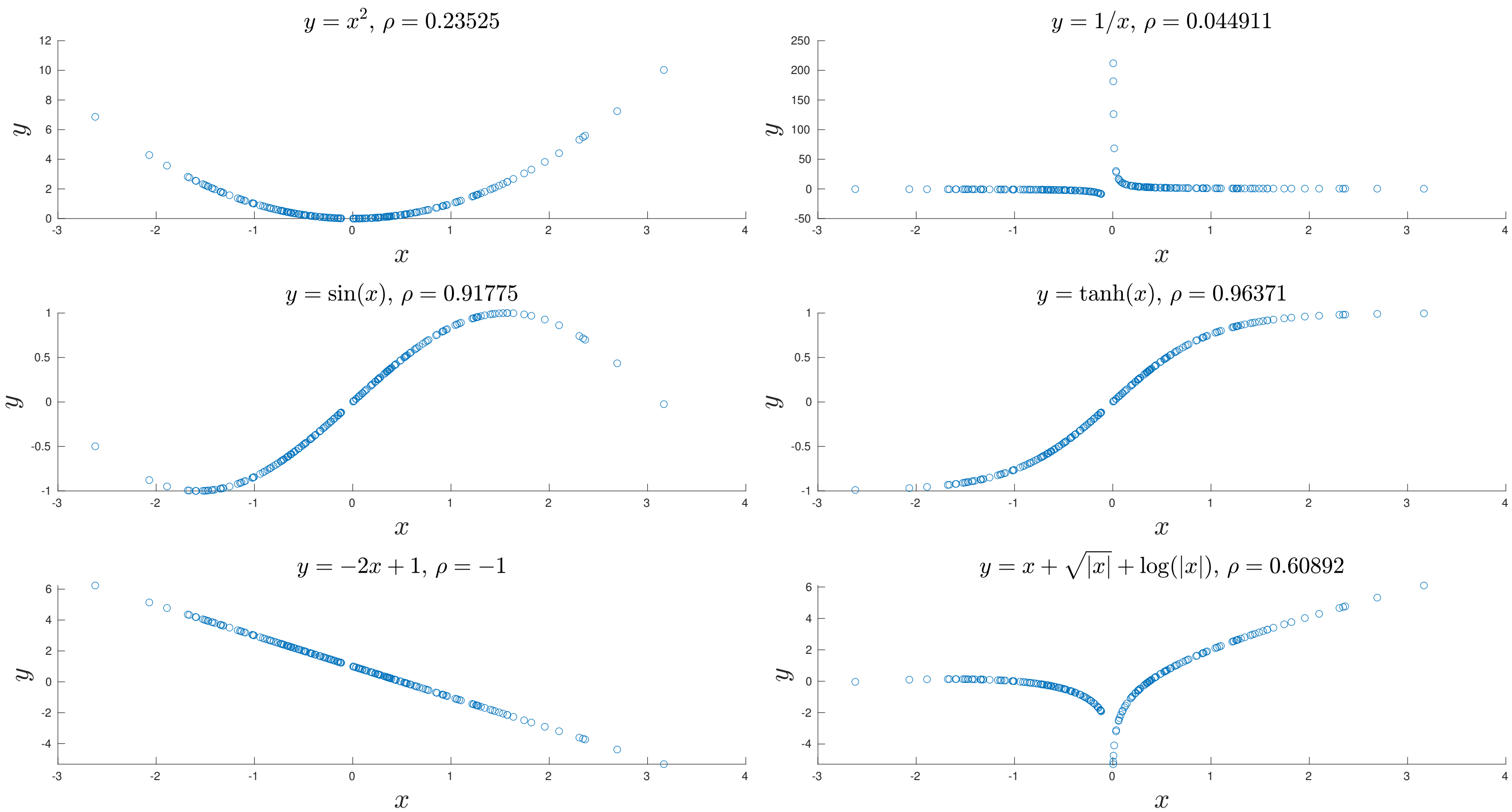
- $-1 \leq \rho_{XY} \leq 1$
- ρ_{XY} gives the linear dependence between X and Y: for $Y = aX + b$,

$$\rho_{XY} = 1 \quad \text{if } a > 0 \quad \text{and} \quad \rho_{XY} = -1 \quad \text{if } a < 0$$

- X and Y are said to be **uncorrelated** if $\rho_{XY} = 0$

Numerical examples of correlation

examples of pairs (x, y) from nonlinear functions



data generation: $X \sim \mathcal{N}(0, 1)$ and $Y = f(X)$; ρ is empirically computed

it is also possible to compute theoretical values of ρ

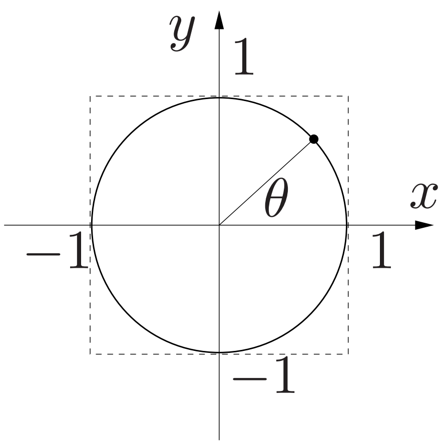
if X and Y are *independent* then X and Y are *uncorrelated*

but the converse is NOT true

example: uncorrelated but dependent random variables

let θ be a uniform RV in the interval $(0, 2\pi)$ and let

$$X = \cos\theta, \quad Y = \sin\theta$$



- the marginals of X and Y are arcsine pdf's
- the products of the marginals of X and Y is nonzero in the square region
- (X, Y) is the point on the unit circle, so they are are dependent

$$\mathbf{E}[XY] = \frac{1}{2\pi}\int_0^{2\pi} \sin\phi\cos\phi \; d\phi = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\phi \; d\phi = 0$$

since $\mathbf{E}[X] = \mathbf{E}[Y] = 0$, the above eq. implies X and Y are uncorrelated

Joint Characteristic Function

the joint characteristic function of X and Y is defined by

$$\Phi_{XY}(\lambda, \omega) = \mathbf{E}[e^{\mathrm{i}(\lambda X + \omega Y)}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\mathrm{i}(\lambda X + \omega Y)} f_{XY}(x, y)\, dx\, dy$$

the joint characterestic function is a 2D Fourier transform

if X and Y are independent

$$\Phi_{XY}(\lambda, \omega) = \mathbf{E}[e^{\mathrm{i}\lambda X}]\mathbf{E}[e^{\mathrm{i}\omega Y}] = \Phi_X(\lambda)\Phi_Y(\omega)$$

Example

let $U \sim \mathcal{N}(0,1)$ and $V \sim \mathcal{N}(0,1)$ be independent RVs

define

$$X = U + V, \quad Y = U - V$$

the joint characteristic function of X, Y is obtained by

$$\begin{aligned}
\Phi_{XY}(\lambda,\omega) &= \mathbf{E}[e^{\mathrm{i}(\lambda(U+V)+\omega(U-V))}] \\
&= \mathbf{E}[e^{\mathrm{i}(\lambda+\omega)U+\mathrm{i}(\lambda-\omega)V}] \\
&= \mathbf{E}[e^{\mathrm{i}(\lambda+\omega)U}]\mathbf{E}[e^{\mathrm{i}(\lambda-\omega)V}] \\
&= \Phi_U(\lambda+\omega)\Phi_V(\lambda-\omega) \\
&= e^{-(\lambda+\omega)^2/2}e^{-(\lambda-\omega)^2/2} \\
&= = e^{-(\lambda^2+\omega^2)} = \Phi_X(\lambda)\Phi_Y(\omega)
\end{aligned}$$

X and Y are also Gaussian with zero mean and variance 2

from the identity:

$$\frac{\partial^2 \mathbf{E}[e^{\mathrm{i}(\lambda X + \omega Y)}]}{\partial\lambda\partial\omega} = \mathrm{i}^2 \mathbf{E}[XYe^{\mathrm{i}(\lambda X + \omega Y)}]$$

the joint characteristic function is also useful for finding $\mathbf{E}[XY]$, since

$$\begin{aligned}
\mathbf{E}[XY] &= \frac{1}{\mathrm{i}^2} \frac{\partial^2 \mathbf{E}[e^{\mathrm{i}(\lambda X + \omega Y)}]}{\partial\lambda\partial\omega}\bigg|_{\lambda=0,\omega=0} \\
&= -\frac{\partial^2 \mathbf{E}[e^{-(\lambda^2+\omega^2)}]}{\partial\lambda\partial\omega}\bigg|_{\lambda=0,\omega=0} \\
&= -e^{-(\lambda^2+\omega^2)}(4\lambda\omega)|_{\lambda=0,\omega=0} \\
&= 0
\end{aligned}$$

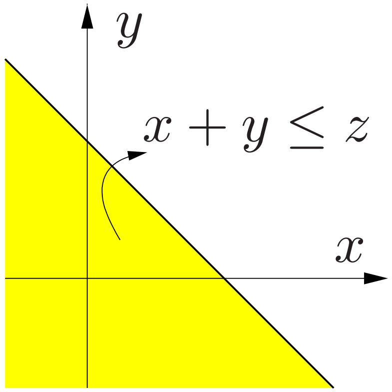
thus X and Y are uncorrelated

(note that X and Y have zero mean)

Function of Multiple Random Variables

- sum of random variables: $Z = X + Y$
- division of random variables: $Z = X/Y$
- linear transformation

Sum of Random Variables



Let $Z = X + Y$

$$P(Z \leq z) = P(X + Y \leq z)$$

integrate the joint pdf f_{XY} over the yellow region

CDF of Z: $F_Z(z) = P(Z \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y) dy dx$

PDF of Z: $f_Z(z) = \dfrac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} f_{XY}(x, z - x) dx$

when X, Y are independent, the pdf of Z has a form of convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

Example

find the pdf of the sum $Z = X + Y$

X, Y are jointly Gaussian with zero mean and unit variance with correlation coefficient $\rho = -1/2$

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(x, z - x)dx \\
&= \frac{1}{2\pi\sqrt{(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-(x^2 - 2\rho x(z-x) + (z-x)^2)/2(1-\rho^2)} dx \\
&= \frac{1}{2\pi\sqrt{3/4}} \int_{-\infty}^{\infty} e^{-(x^2 - xz + z^2)/2(3/4)} dx \\
&= \frac{e^{-z^2/2}}{\sqrt{2\pi}}
\end{aligned}$$

the sum of these two *nonindependent* Gaussian is also a Gaussian RV

Characteristic function of a sum

let X and Y be *independent* RVs and define

$$Z = X + Y$$

then CF of X is the *product* of CFs of X and Y:

$$\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega)$$

proof.

$$\begin{aligned} \Phi_Z(\omega) &= \mathbf{E}[e^{\mathrm{i}\omega Z}] = \mathbf{E}[e^{\mathrm{i}\omega(X+Y)}] \\ &= \mathbf{E}[e^{\mathrm{i}\omega X}]\,\mathbf{E}[e^{\mathrm{i}\omega}Y] \quad (\because X \text{ and } Y \text{ are independent}) \\ &= \Phi_X(\omega)\,\Phi_Y(\omega) \end{aligned}$$

Example: sum of independent binomials

let X and Y be i.i.d. binomials RVs with parameters n, p

$$P_X(k) = P_Y(k) = \left(\begin{array}{c} n \\ k \end{array}\right) p^k\, q^{n-k} \qquad (q = 1 - p)$$

first compute the CF of X and Y

$$\Phi_X(\omega) = \Phi_Y(\omega) = \sum_{k=0}^{n} e^{\mathrm{i}\omega k} \left(\begin{array}{c} n \\ k \end{array}\right) p^k\, q^{n-k} = (pe^{\mathrm{i}\omega} + q)^n$$

the CF of Z is then given by

$$\Phi_Z(\omega) = \Phi_X(\omega)\, \Phi_Y(\omega) = (pe^{\mathrm{i}\omega} + q)^{2n}$$

conclusion: Z is also a binomial with parameters $2n$ and p

Division of Random Variables

let $Z = X/Y$

if $Y = y$ (given), then $Z = X/y$, a scaled version of X

therefore, if Y is fixed then the distribution of Z must be the same as X

$$f_{Z|Y}(z|y) = |y| f_{X|Y}(yz|y)$$

use this result to find the pdf of Z:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} |y| f_{X|Y}(yz|y) f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} |y| f_{XY}(yz, y)\, dy \end{aligned}$$

Division of Exponential RVs

let X and Y be exponential RVs with mean 1

$$f_X(x) = e^{-x},\ x \geq 0, \qquad f_Y(y) = e^{-y},\ y \geq 0$$

assume that X, Y are independent, so

$$f_{XY}(x,y) = f_X(x) f_Y(y) = e^{-(x+y)}$$

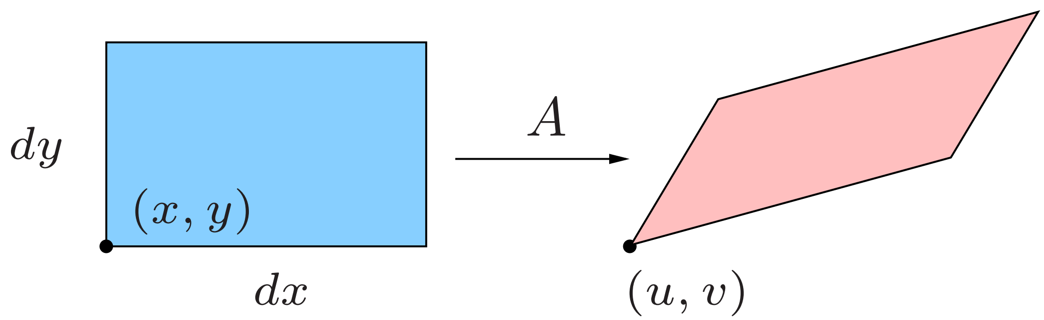
the pdf of $Z = Y/X$ can be determined by

$$f_Z(z) = \int_0^\infty y e^{-yz} e^{-y} dy = \frac{1}{(z+1)^2}, \quad z > 0$$

Linear Transformation

let A be an *invertible* linear transformation such that

$$\begin{bmatrix} U \\ V \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} X \\ Y \end{bmatrix} = A^{-1} \begin{bmatrix} U \\ V \end{bmatrix}$$



dM:

area of the parallelogram

$$P(X \in dx, Y \in dy) = f_{XY}(x,y)dxdy, \quad P(U \in du, V \in dV) = f_{UV}(u,v)dM$$

it can be shown that

$$\begin{aligned} dM &= |\det A| dx\, dy, \\ f_{UV}(u,v) &= \frac{1}{|\det A|} f_{XY}(x,y) \end{aligned}$$

Example: Linear Transformation of a Gaussian

let X and Y be jointly Gaussian RVs with the joint pdf

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{3/4}} \text{ exp } -\frac{2(x^2 - xy + y^2)}{3}$$

let U and V be obtained from (X, Y) by

$$\begin{bmatrix} U \\ V \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

therefore the pdf of U and V is

$$f_{UV}(u,v) = \frac{1}{\pi\sqrt{3}} \text{ exp } -(u^2/3 + v^2)$$

U and V become independent, zero-mean Gaussian RVs

References

Chapter 5 in
A. Leon-Garcia, *Probability, Statistics, and Random Processes for Electrical Engineering*, 3rd edition, Pearson Prentice Hall, 2009