2. Reviews on dynamical systems

- linear systems: state-space equations
- random (stochastic) processes

Continous-time systems

- an autonomous system

$$\dot{x}(t) = Ax(t), \quad y = Cx(t)$$

- a system with inputs

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y = Cx(t) + Du(t)$$

- $x \in \mathbf{R}^n$ is the state, $y \in \mathbf{R}^m$ is the output, and $u \in \mathbf{R}^p$ is the control input
- $A \in \mathbf{R}^{n \times n}$ is the dynamic matrix
- $B \in \mathbf{R}^{p \times n}$ is the input matrix
- $C \in \mathbf{R}^{m \times n}$ is the output matrix
- $D \in \mathbf{R}^{m \times p}$ is the direct forward term

Solution of state-space equations

- an autonomous system

$$x(t) = e^{At}x(0), \quad y = Ce^{At}x(0)$$

e^{At} is the state-transition matrix; can be computed analytically

- a system with inputs

$$\begin{aligned} x(t) &= e^{tA}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \\ y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

$x(t)$ consists of zero-input response and zero-state response

Discrete-time systems

- an autonomous system

$$x(t+1) = Ax(t), \quad y(t) = Cx(t)$$

with solution

$$x(t) = A^t x(0)$$

- a system with inputs

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with solution

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau)$$

Transfer function of linear systems

explains a relationship from u to y

- continuous-time system: $Y(s) = H(s)U(s)$

$$H(s) = C(sI - A)^{-1}B + D$$

- discrete-time system: $Y(z) = H(z)U(z)$

$$H(z) = C(zI - A)^{-1}B + D$$

the inverse Laplace (z-) transform of H is the impulse response, $h(t)$

Important concepts of system analysis

- stability: if $x(t) \to 0$ when $t \to \infty$
  (eigenvalues of dynamic matrix, Lyapunov theory)

- controllability: how a target state can be acheived by applying a certain input
  (explained from A and B)

- observability: how to estimate $x(0)$ from the measurement y
  (explained A and C)

Stochastic Signals

- stationary processes
- ergodic processes
- correlation and covariance function
- power spectral density
- independent and uncorrelated processes
- Gaussian or normal processes
- white noise
- linear process with stochastic signals

Stochastic Processes

stochastic process is an entirely family (*ensemble*) of random time signals

$$\{x(t), \quad t \in T\}$$

i.e., for each t in the index set T, $x(t)$ is a random variable

- a signal realization $x(t)$ is called *sample function* or a *sample path*
- if T is a countable set, $x(t)$ is called **discrete-time** stochastic process
- if T is a continuum, $x(t)$ is called **continuous-time** process
- a process can be either discrete- or continuous-valued

Joint distribution

let $x_1, \ldots, x_n$ be the n random variables by sampling the process $x(t)$

$$x_1 = x(t_1), \;\; x_2 = x(t_2), \ldots, \;\; x_n = x(t_n)$$

a stochastic process is specified by the collection of joint cdf (depend on time)

$$F(x_1, x_2, \ldots, x_n) = P(x(t_1) \leq x_1, \;\; x(t_2) \leq x_2, \ldots, \;\; x(t_n) \leq x_n)$$

- continuous-valued process:

$$f(x_1, \ldots, x_n) dx_1 \cdots dx_n = \\ P(x_1 < x(t_1) \leq x_1 + dx_1, \ldots, x_n < x(t_n) \leq x_n + dx_n)$$

- discrete-valued process:

$$p(x_1, x_2, \ldots, x_n) = P(x(t_1) = x_1, x(t_2) = x_2, \ldots, x(t_n) = x_n)$$

Mean and variance of stochastic process

mean and variance function of a continous-time process are defined by

$$\begin{aligned}\mu(t) &= \mathbf{E}[x(t)] = \int_{-\infty}^{\infty} x f(x) dx \\ \mathbf{var}[x(t)] &= \int_{-\infty}^{\infty} (x - \mu(t))^2 f(x) dx\end{aligned}$$

- here f is the pdf of $x(t)$ (depend on time)
- mean and variance are *deterministic* functions of time

Correlation and Covariance

suppose X, Y are random variables with means μ_x and μ_y respectively

cross correlation

$$R_{xy} = \mathbf{E}[XY^T]$$

autocorrelation

$$R = \mathbf{E}[XX^T]$$

cross covariance

$$C_{xy} = \mathbf{E}\left[(X - \mu_x)(Y - \mu_y)^T\right]$$

autocovariance

$$C = \mathbf{E}\left[(X - \mu_x)(X - \mu_x)^T\right]$$

correlation = covariance when considering zero mean

Correlation and Covariance functions

suppose $x(t), y(t)$ are random processes

cross correlation

$$R_{xy}(t_1, t_2) = \mathbf{E}x(t_1)y(t_2)^T$$

autocorrelation

$$R(t_1, t_2) = \mathbf{E}x(t_1)x(t_2)^T$$

cross covariance

$$C_{xy}(t_1, t_2) = \mathbf{E}\left[(x(t_1) - \mu_x(t_1))(y(t_2) - \mu_y(t_2))^T\right]$$

where $\mu_x(t) = \mathbf{E}x(t)$ and $\mu_y(t) = \mathbf{E}y(t)$

autocovariance

$$C(t_1, t_2) = \mathbf{E}\left[(x(t_1) - \mu(t_1))(x(t_2) - \mu(t_2))^T\right]$$

Stationary processes

a process is called **strictly stationary** if the joint cdf of

$$x(t_1), x(t_2), \ldots, x(t_n)$$

is *the same as* that of

$$x(t_1 + \tau), x(t_2 + \tau), \ldots, x(t_n + \tau)$$

for *all time shifts* τ and for *all choices of sample times* $t_1, \ldots, t_k$

- first-order cdf of a stationary process must be independent of time

$$F_{x(t)}(x) = F_{x(t+\tau)}(x) = F(x), \quad \forall t, \tau$$

  implication: mean and variance are **constant** and **independent** of time

Wide-sense stationary Process

a process is **wide-sense** stationary if the two conditions hold:

1. $\mathbf{E}[x(t)] =$ constant for all t
2. $R(t_1, t_2) = R(t_1 - t_2)$ (only depends on the time gap)

the correlation/covariance functions are simplified to

$$R(\tau) = \mathbf{E}x(t+\tau)x(t)^T, \qquad R_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T$$

$$C(\tau) = \mathbf{E}x(t+\tau)x(t)^T - \mu_x\mu_x^T, \qquad C_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T - \mu_x\mu_y^T$$

Example

determine the mean and the autocorrelation of a random process

$$x(t) = A\cos(\omega t + \phi)$$

where the random variables A and ϕ are independent and ϕ is uniform on $(-\pi, \pi)$

since A and ϕ are independent, the mean is given by

$$\mathbf{E}x(t) = \mathbf{E}[A]\mathbf{E}[\cos(\omega t + \phi)]$$

using the uniform distribution in ϕ, the last term is

$$\mathbf{E}\cos(\omega t + \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t + \phi)d\phi = 0$$

therefore, $\mathbf{E}x(t) = 0$

using trigonometric identities, the autocorrelation is determined by

$$\mathbf{E}x(t_1)x(t_2) = \frac{1}{2}\mathbf{E}A^2\mathbf{E}[\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\phi)]$$

since

$$\mathbf{E}[\cos(\omega t_1 + \omega t_2 + 2\phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 + 2\phi)d\phi = 0$$

we have

$$R(t_1, t_2) = (1/2)\mathbf{E}[A^2]\cos\omega(t_1 - t_2)$$

hence, the random process in this example is wide-sense stationary

Power Spectral Density

Wiener-Khinchin Theorem: if a process is wide-sense stationary, the autocorrelation function and the power spectral density form a Fourier transform pair:

$$S(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega\tau} R(\tau) d\tau \qquad \text{continuous}$$

$$S(\omega) = \sum_{k=-\infty}^{k=\infty} R(k) e^{-\mathrm{i}\omega k} \qquad \text{discrete}$$

the autocorrelation function at $\tau = 0$ indicates the average power:

$$R(0) = \mathbf{E}[x(t)x(t)^T] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega$$

(similarly, use discrete inverse Fourier transform for discrete systems)

Properties

- $R(-t) = R(t)^T$ (if the process is scalar, then $R(-t) = R(t)$)
- non-negativity: that is for any $a_i, a_j \in \mathbf{R}^n$, with $i, j = 1, \dots N$, we have

$$\sum_i^N \sum_j^N a_i^T R(i-j) a_j \geq 0,$$

  which follows from

$$\sum_i^N \sum_j^N a_i^T R(i-j) a_j = \sum_i^N \sum_j^N \mathbf{E}[a_i^T x(i) x(j)^T a_j] = \mathbf{E}\left[\left(\sum_i^N a_i^T x(i)\right)^2\right] \geq 0.$$

- $S(\omega)$ is self-adjoint, *i.e.*, $S(\omega)^* = S(\omega)$ for all ω
- diagonals of $S(\omega)$ are real-valued

Ergodic Processes

a stochastic process is *ergodic* if

$$\mathbf{E}[x(t)] = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt \qquad \text{(continuous)}$$

$$\mathbf{E}[x(t)] = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x(k) \qquad \text{(discrete)}$$

(time average $=$ ensemble average)

- one typically gets statistical information from emsemble averaging
- ergodic hypothesis means this information can also be obtained from averaging a single sample $x(t)$ over *time*

with ergodic assumption,

continous time

$$
\begin{aligned}
R(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)x(t)^T dt \\
R_{xy}(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t+\tau)y(t)^T dt
\end{aligned}
$$

discrete time

$$
\begin{aligned}
R(\tau) &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x(k+\tau)x(k)^T \\
R_{xy}(\tau) &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x(k+\tau)y(k)^T
\end{aligned}
$$

Independent and Correlated Processes

stationary processes $x(t)$ and $y(t)$ are called **independent** if

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

(the joint pdf is equal to the product of marginals)

and are called **uncorrelated** if

$$C_{xy}(\tau) = 0, \quad \forall \tau$$

- independent processes are always uncorrelated
- the opposite may not be true

White noise

a zero-mean process with the following properties:

continuous time

$$R(\tau) = S_0\delta(\tau), \qquad S(\omega) = \int_{-\infty}^{\infty} S_0\delta(\tau)e^{-i\omega\tau}\,d\tau = S_0$$

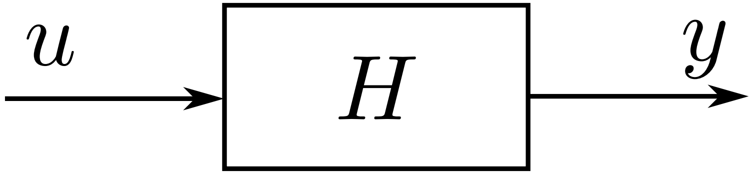
discrete time

$$R(k) = S_0\delta(k) = \begin{cases} S_0, & k = 0 \\ 0, & k \neq 0 \end{cases}, \qquad S(\omega) = \sum_{k=-\infty}^{\infty} S_0\delta(k)e^{-i\omega k} = S_0$$

(constant spectrum)

Linear systems with random input

let y be the response to input u under a linear causal system H



Facts: if $u(t)$ is a wide-sense stationary process and H is stable then

- $y(t)$ is also a wide-sense stationary process
- spectrum of u and y are related by

$$S_y(\omega) = H(\omega)S_u(\omega)H(\omega)^*$$

where $H(\omega)^*$ is the complex conjugate transpose of $H(\omega)$

Output covariance of linear system

given a system: $x(t+1) = Ax(t) + Bw(t)$; $w(t)$ is white noise with covariance Σ_w

if A is **stable** and $x(0)$ is uncorrelated with $w(t)$ for all $t \geq 0$ then

- $\lim_{t\to\infty} \mathbf{E}[x(t)] = 0$ and the autocovariance of x converges to $\Sigma \succ 0$

$$\lim_{t\to\infty} C(t,t) = \Sigma$$

where Σ is a solution to the **Lyapunov** equation

$$\Sigma = A\Sigma A^T + B\Sigma_w B^T$$

- $x(t)$ is a wide-sense stationary process in **steady-state**

$$\lim_{t\to\infty} C(t+\tau, t) = C(\tau) = \begin{cases} A^\tau \Sigma, & \tau \geq 0 \\ \Sigma (A^T)^{|\tau|}, & \tau < 0 \end{cases}$$

Random walk

a process $x(t)$ is a random walk if

$$x(t) = x(t-1) + w(t-1)$$

where $w(t)$ is a white noise with covariance Σ

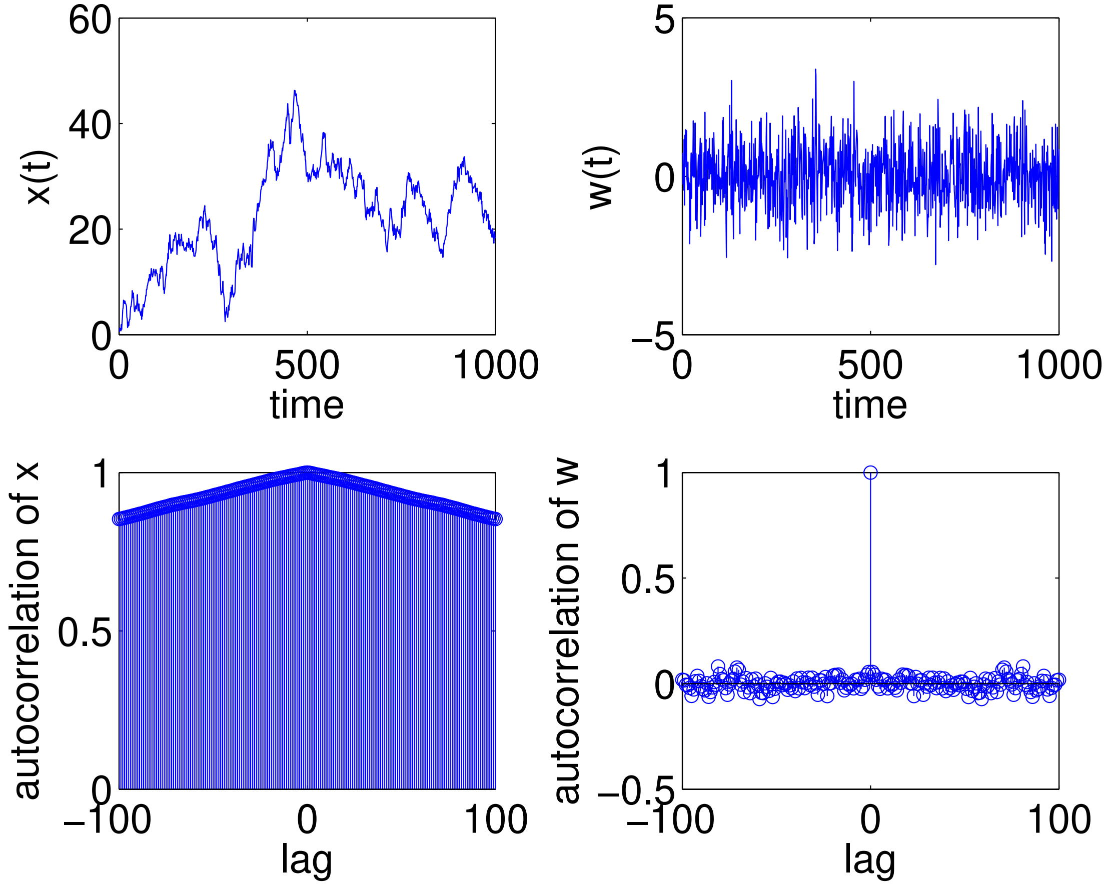
- $x(t)$ obeys a linear (unstable) system with a random input
- with back substitution, we can express $x(t)$ as

$$x(t) = w(t-1) + w(t-2) + \cdots + w(0)$$

- $x(t)$ is *non-stationary* because $R(t, t+\tau)$ depends on t

$$R(t, t+\tau) = \mathbf{E}[x(t)x(t+\tau)^T] = t\Sigma$$

time plot of random walk and its normalized *sample* autocorrelation (correlogram)



correlogram of x gradually decays

References

Chapter 9 in

A. Leon-Garcia, *Probaility, Statistics, and Random Processes for Electrical Engineering*, 3rd edition, Pearson, 2009

Chapter 3 in
T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989