13. Model selection and model validation

- model selection aspects
- bias and variance
- model selection: criterions, cross-validation
- model validation: whiteness test, cross-correlation test

Factors in model selection

objective: obtain a good model at a low cost

1. **quality of the model:** defined by a measure of the goodness, e.g., the mean-squared error, log-likelihood
   - MSE consists of a *bias* and a *variance* contribution
   - a complex model has small bias but higher variance (than a simple model)
   - the best model structure is therefore a trade-off between *flexibility* and *parsimony*
2. **price of the model:** an estimation method (which typically results in an optimization problem) highly depends on the model structures, which influences:
   - algorithm complexity
   - properties of the loss function
3. **intended use of the model:** prediction, controller design, inference

Bias-variance decomposition

assume that the observation Y obeys

$$Y = f(X) + \nu, \quad \mathbf{E}\nu = 0, \quad \mathbf{cov}(\nu) = \sigma^2$$

the mean-squared error of a regression fit $\hat{f}(X)$ at $X = x_0$ is

$$\begin{aligned}\mathsf{MSE} &= \mathbf{E}[(Y - \hat{f}(x_0))^2 | X = x_0] \\ &= \sigma^2 + [\mathbf{E}\hat{f}(x_0) - f(x_0)]^2 + \mathbf{E}[\hat{f}(x_0) - \mathbf{E}\hat{f}(x_0)]^2 \\ &= \sigma^2 + \mathsf{Bias}^2(\hat{f}(x_0)) + \mathsf{Var}(\hat{f}(x_0))\end{aligned}$$

- this relation is known as **bias-variance decomposition**
- no matter how well we estimate $f(x_0)$, σ^2 represents *irreducible error*
- typically, the more complex we make model $\hat{f}$, the lower the bias, but the higher the variance

proof of bias-variance decomposition: note that

- the true f is deterministic
- $\mathbf{var}(Y|X=x) = \sigma^2$ and $\mathbf{E}[Y|X=x] = f(x)$
- $\hat{f}(x)$ is random

we will omit the notation of conditioning on $X = x$

$$\begin{aligned}
\mathbf{E}[(Y - \hat{f}(X))^2] &= \mathbf{E}[Y^2] + \mathbf{E}[\hat{f}(x)^2] - \mathbf{E}[2Y\hat{f}(x)] \\
&= \mathbf{var}(Y) + \mathbf{E}[Y]^2 + \mathbf{var}\,\hat{f}(x) + \mathbf{E}[\hat{f}(x)]^2 - 2f(x)\mathbf{E}[\hat{f}(x)] \\
&= \mathbf{var}(Y) + f(x)^2 + \mathbf{var}\,\hat{f}(x) + \mathbf{E}[\hat{f}(x)]^2 - 2f(x)\mathbf{E}[\hat{f}(x)] \\
&= \sigma^2 + \mathbf{var}\,\hat{f}(x) + (f(x) - \mathbf{E}[\hat{f}(x)])^2 \\
&= \sigma^2 + \mathbf{var}\,\hat{f}(x) + (\mathbf{E}[f(x) - \hat{f}(x)])^2 \\
&= \sigma^2 + \mathbf{var}\,\hat{f}(x) + [\mathrm{Bias}(\hat{f}(x))]^2
\end{aligned}$$

Bias and variance in linear models

two nested linear regression models: predictor X in $\mathcal{M}_1$ is also contained in $\mathcal{M}_2$

$$\mathcal{M}_1 : y = X\beta \quad \text{VS} \quad \mathcal{M}_2 : y = \begin{bmatrix} X & \tilde{x} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \triangleq Z\gamma$$

setting: two models are estimated by LS method, denoted by $\hat{\beta}$ and $\hat{\gamma}$

1. $\mathcal{M}_2$ has lower MSE in predicting y than the MSE of $\mathcal{M}_1$
2. $\mathbf{cov}(\hat{\beta})$ of $\mathcal{M}_2$ is larger than $\mathbf{cov}(\hat{\beta})$ of $\mathcal{M}_1$
3. variance of $\hat{y}$ from $\mathcal{M}_2$ is higher than that of $\mathcal{M}_1$

$\mathcal{M}_2$ (complex model) has less bias but more variance both in estimator and prediction

our proof will use subscript 1 for $\mathcal{M}_1$ and and 2 for $\mathcal{M}_2$

Inverse of block matrices

the inverse of a block matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \succ 0$$

can be obtained in block using Schur complement: $S = (D - CA^{-1}B)^{-1} \succ 0$

$$X^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \quad (1)$$

we often encounter the difference of two quadratic forms

$$\begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} - u^T A^{-1} u = (v - B^T A^{-1} u)^T S^{-1} (v - B^T A^{-1} u) \geq 0 \quad (2)$$

which is always non-negative

proof of $\mathbf{MSE}_2 \leq \mathbf{MSE}_1$

- let P_1 and P_2 be orthogonal projection of y onto $\mathcal{R}(X)$ and $\mathcal{R}(Z)$, resp
- it can be shown that $\mathrm{MSE}_1 = \|y\|_2^2 - y^T P_1 y$ and $\mathrm{MSE}_2 = \|y\|_2^2 - y^T P_2 y$
- it is left to show that $y^T P_2 y \geq y^T P_1 y$

$$P_2 = Z(Z^T Z)^{-1} Z^T = \begin{bmatrix} X & \tilde{x} \end{bmatrix} \begin{bmatrix} X^T X & X^T \tilde{x} \\ \tilde{x}^T X & \tilde{x}^T \tilde{x} \end{bmatrix}^{-1} \begin{bmatrix} X^T \\ \tilde{x}^T \end{bmatrix}, \quad P_1 = X(X^T X)^{-1} X^T$$

- apply the inverse of block matrix

$$P_2 - P_1 = (\tilde{x} - X(X^T X)^{-1} X^T \tilde{x}) S^{-1} (\tilde{x} - X(X^T X)^{-1} X^T \tilde{x})^T \succeq 0$$

where $S = \tilde{x}^T \tilde{x} - \tilde{x}^T X (X^T X)^{-1} X^T \tilde{x}$

proof of $\mathbf{cov}(\hat{\beta}_2) \succeq \mathbf{cov}(\hat{\beta}_1)$

- $\mathbf{cov}(\hat{\beta}_2)$ is the leading (1,1) block of $\mathbf{cov}(\hat{\gamma})$, while $\mathbf{cov}(\hat{\beta}_1) = (X^T X)^{-1}$
- use $\mathbf{cov}(\hat{\gamma}) = (Z^T Z)^{-1}$ and the inverse of block matrix

$$(Z^T Z)^{-1} = \begin{bmatrix} X^T X & X^T \tilde{x} \\ \tilde{x}^T X & \tilde{x}^T \tilde{x} \end{bmatrix}^{-1} \triangleq \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1} B S^{-1} B^T A^{-1} & \times \\ \times & \times \end{bmatrix}$$

where $S = D - B^T A^{-1} B \succeq 0$

- $\mathbf{cov}(\hat{\beta}_2)$ is bigger than $\mathbf{cov}(\hat{\beta}_1)$ because

$$\mathbf{cov}(\hat{\beta}_2) - \mathbf{cov}(\hat{\beta}_1) = A^{-1} + A^{-1} B S^{-1} B^T A^{-1} - A^{-1} = A^{-1} B S^{-1} B^T A^{-1} \succeq 0$$

proof of $\mathbf{var}(\hat{y}_2) \geq \mathbf{var}(\hat{y}_1)$

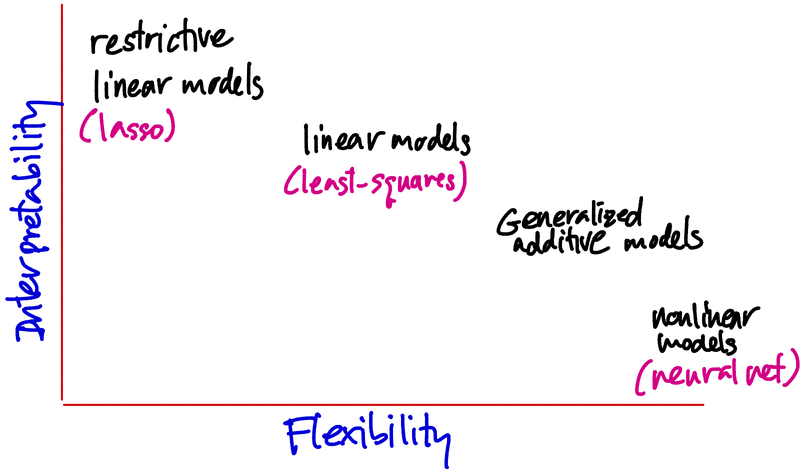
- suppose $\hat{y}_1 = u^T\hat{\beta}$ and $\hat{y}_2 = w^T\hat{\gamma}$ where $w = (u, v)$
- we test prediction of y from new regressors u and (u, v)
- since the model is simply linear, the variance can be obtained by

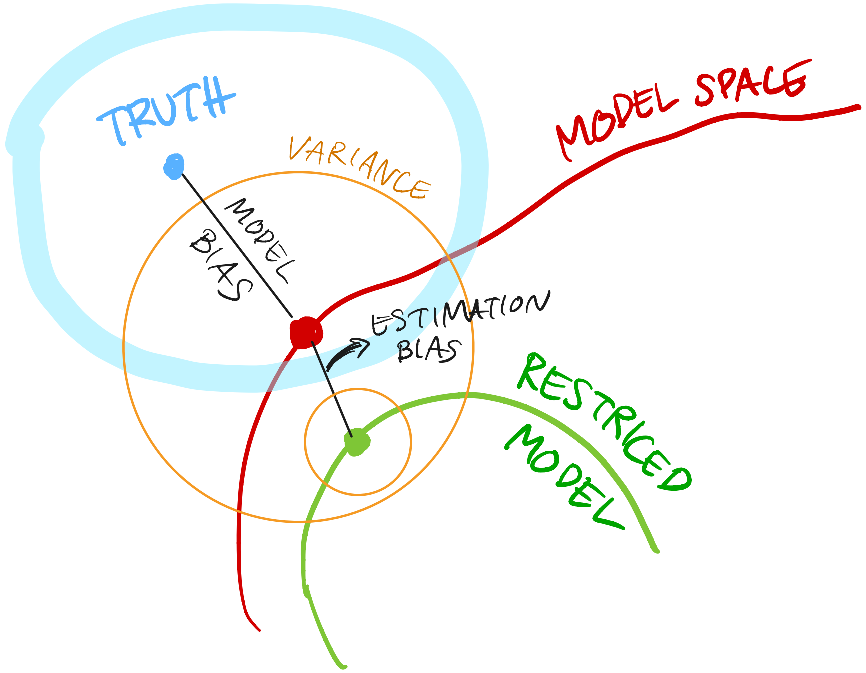
$$\begin{aligned}\mathbf{var}(\hat{y}_2) - \mathbf{var}(\hat{y}_1) &= w^T\,\mathbf{cov}(\gamma)w - u^T\,\mathbf{cov}(\beta)u \\ &= \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} X^TX & X^T\tilde{x} \\ \tilde{x}^TX & \tilde{x}^T\tilde{x} \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} - u^T(X^TX)^{-1}u\end{aligned}$$

- the difference is non-negative (using result on page 13-6)

Model properties

consider bias and variance of model with different structures



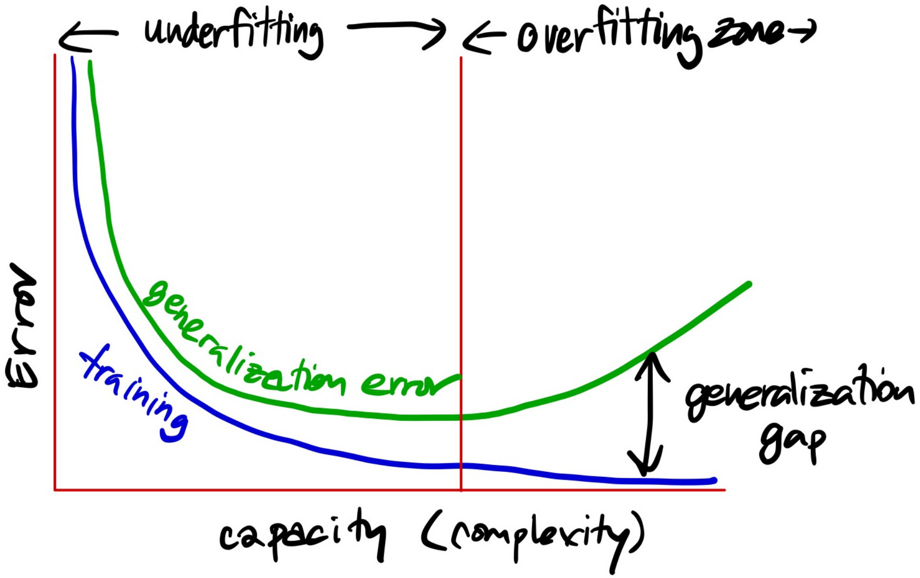


(T. Hastie *et.al. The Elements of Statistical Learning*, Springer, 2010 page 225)

a simple model has less flexibility (more bias) but easy to interpret and has less variance

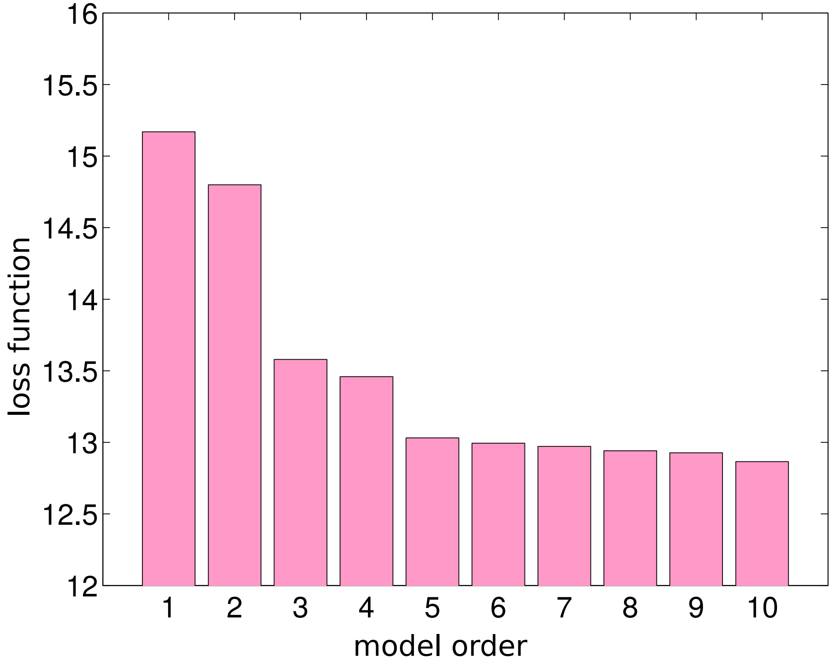
U-shape of generalization error

models are estimated on training data set and evaluated on test set (unseen data)



- training errors always decrease as model complexity increase
- generalization error initially decreases as model picks up relevant features of data
- however, if the model complexity exceeds a certain degree, the generalization error can rise up again – this is when we observe overfitting

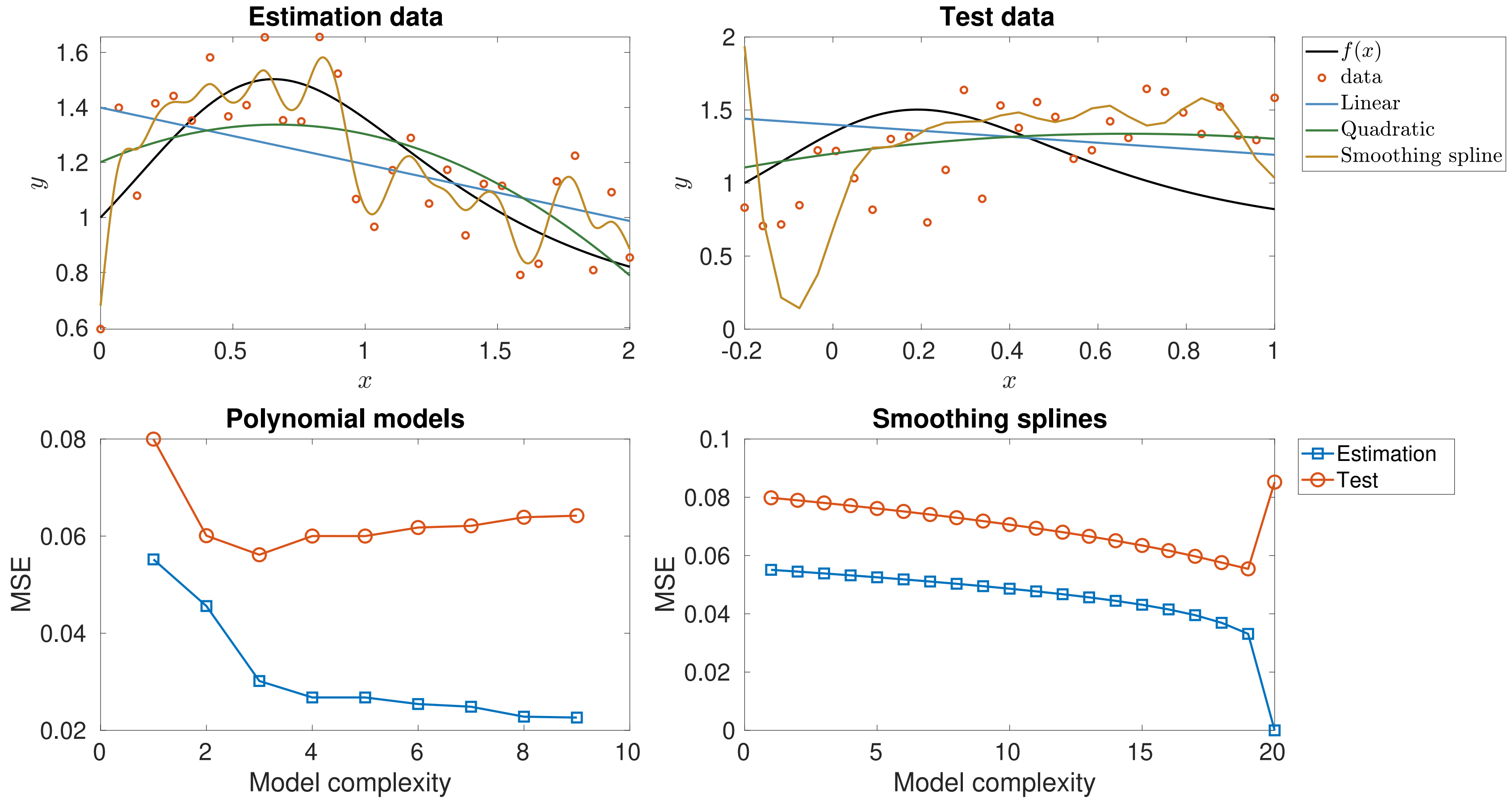
Model fitting versus model complexity



- true AR model has order $p = 5$
- estimate AR model using LS method
- plot loss function (MSE) at different model orders

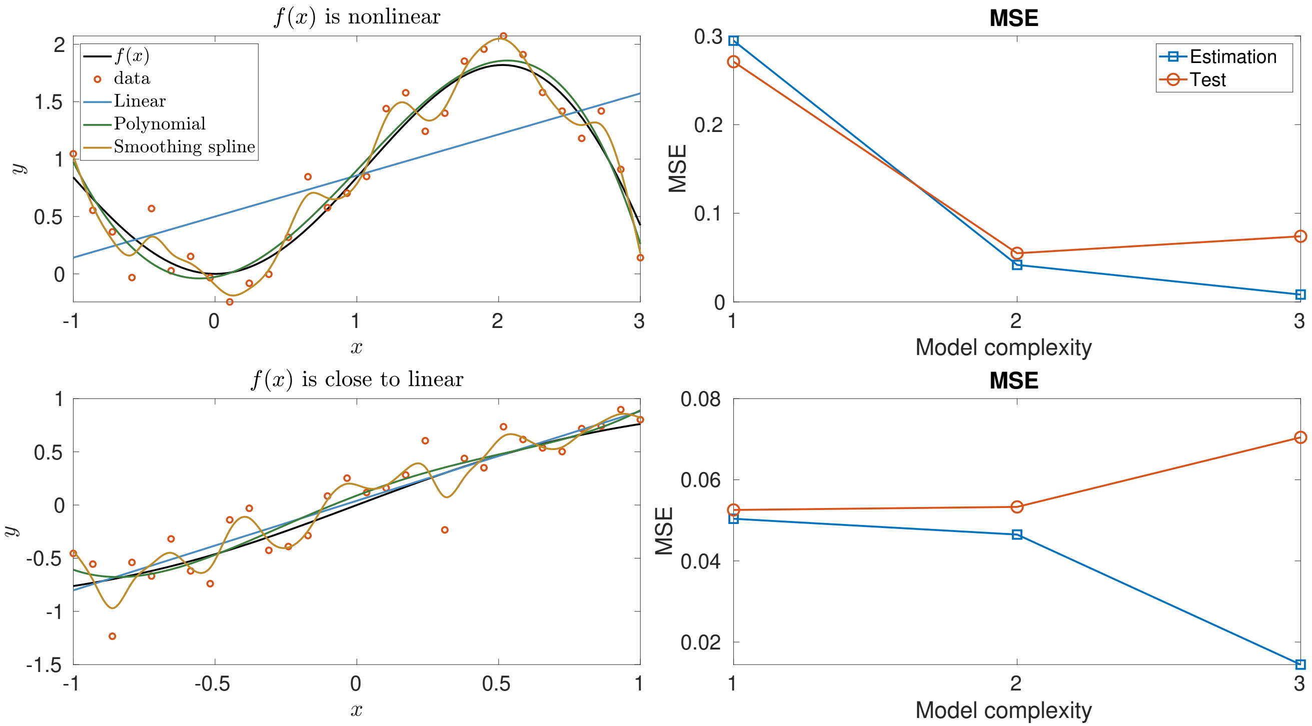
- the minimized loss is a decreasing function of the model order
- loss function begins to decrease as the model picks up the relevant features
- as p increases, the model tends to *over fit* the data
- in practice, we look for the "knee" in the curve (around $p = 5$)

Observe overfitting on test error



- too complex models cannot generalize well on test (unseen) data
- overfitting occurs when MSE on test set decreases but starts to rise again

Does overfitting always occur?



- when the true description is highly nonlinear, test MSE does not significantly increase
- overfitting is apparent when the estimated model is more complex (than it should be) in order to explain a simpler ground-truth model

Model selection

- model selection criterions
- cross-validation

Model selection criterion

parsimony principle: among competing models which all explain the data well, the model with the smallest number of parameters should be chosen

a model selction criterion consists of two parts:

$$\text{loss function} + \text{model complexity}$$

- the first term is to assess the quality of the model, e.g., likelihood function, RSS, MSE, Fit Percent $(1 - \frac{\|y-\hat{y}\|}{\|y-\bar{y}\|}) \times 100\%$
- the second term is to penalize the model order and grows as the number of parameters increases
- we choose the best model as the one with the lowest model selection score

What exactly do we choose in a model?

consider an additive error model

$$y_i = g(x_i;\theta) + e_i, \quad e_i \sim \mathcal{N}(0,\sigma^2), \quad i = 1,2,\ldots,N$$

model selection can be choosing

- a list of predictors x
- a degree of polynomial function g
- a number of basis functions used to decompose g

consider a dynamical model with additive noise (*e.g.*, ARX, FIR)

$$y(t) = g(t, Z^{t-1};\theta) + e(t), \quad e(t) \sim \mathcal{N}(0,\Sigma), \quad t = 1,2,\ldots,N$$

model selection can be choosing order (p,q,r) in ARMA model, or order of FIR let α be a parameter that indicates model complexity

- ARX or FIR orders
- penalty parameter in regularized regression
- the number of predictors in regression models

what can be a function of α ?

- model quality: it indicate the model fitting at such degree of complexity – such as $\mathcal{L}(\alpha), \mathrm{RSS}(\alpha)$
- prediction error: $\varepsilon(t,\theta) = y(t) - \hat{y}(t,\theta)$
- the effective number of parameters (d)

other parameters that involve in model selection scores: N (samples) and output dimension

Model selection scores

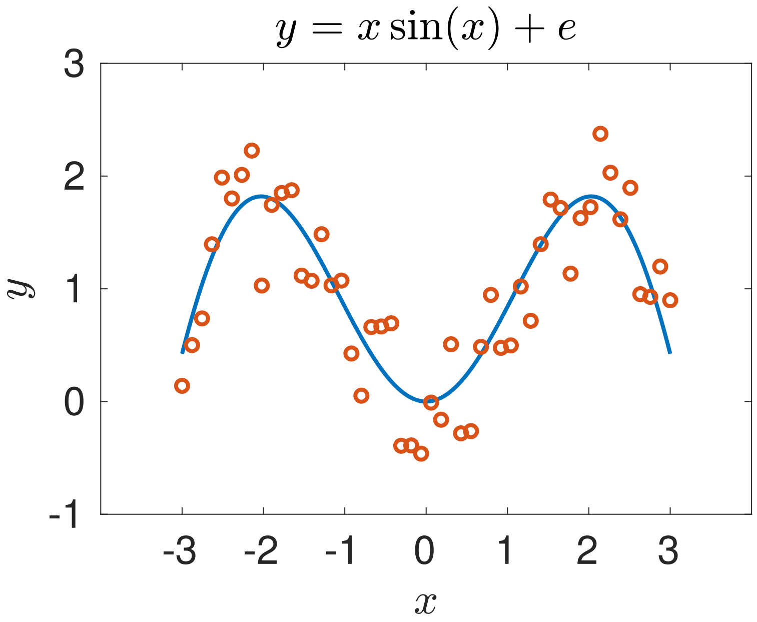
model quality: $\mathcal{L}$: log-likelihood, V: loss function

model complexity: d: effective number of parameters

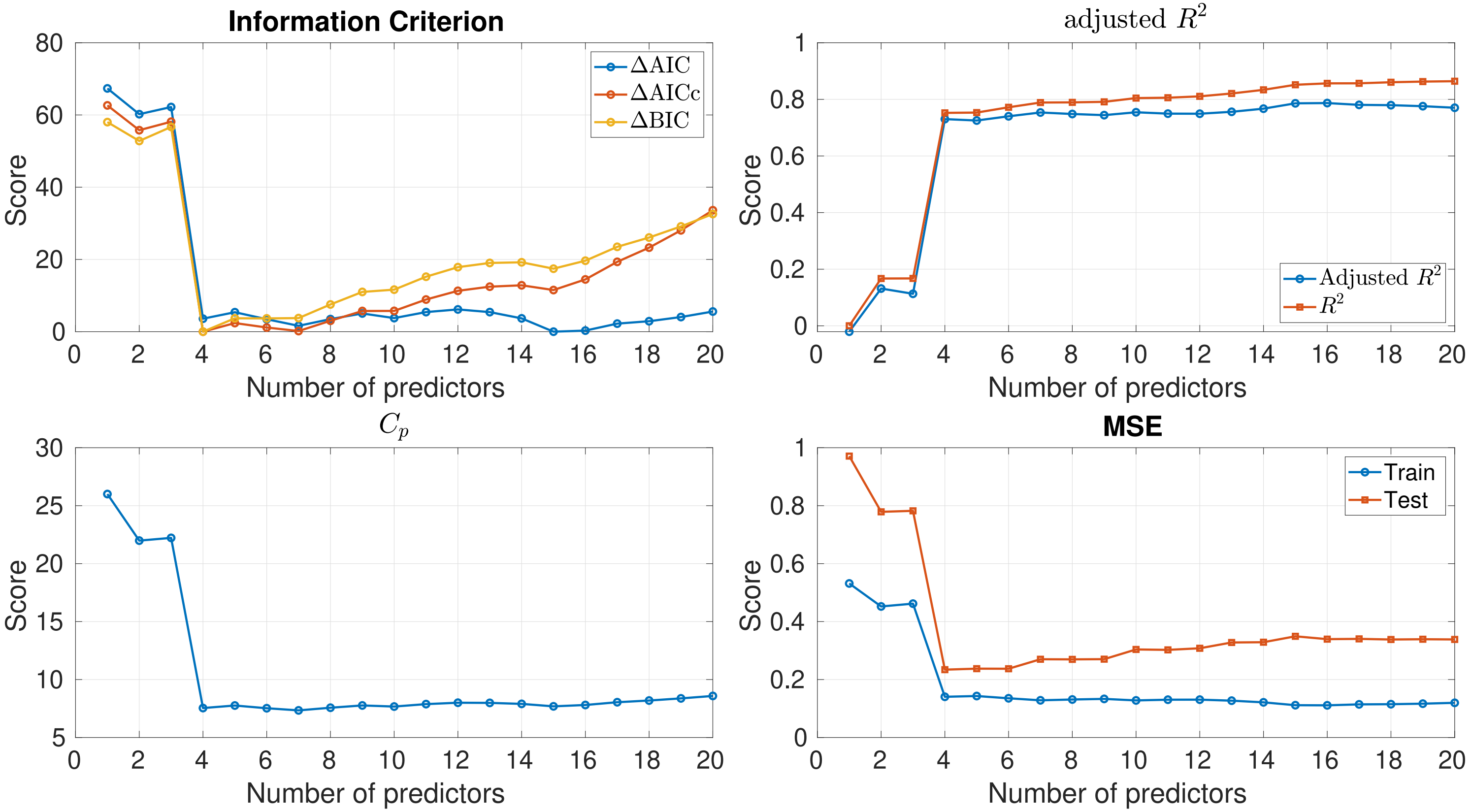
- Akaike information criterion (AIC): AIC$(\alpha) = -2\mathcal{L}(\alpha) + 2d$
- corrected Akaike information (AICc): AICc$(\alpha) = -2\mathcal{L}(\alpha) + 2d + \frac{2d(d+1)}{N-d-1}$
- Bayesian information criterion (BIC): BIC$(\alpha) = -2\mathcal{L}(\alpha) + d \log N$
- Akaike's final prediction-error criterion (FPE): FPE$(\alpha) = V(\hat{\theta}) \left(\frac{1+d/N}{1-d/N}\right)$
- Mallow's C_p: $C_p(\alpha) = \frac{1}{N}\left[\mathrm{RSS}(\alpha) + 2d\hat{\sigma}^2\right]$
- adjusted R^2: $1 - \frac{\mathrm{RSS}(\alpha)/(N-d-1)}{\mathrm{TSS}/(N-1)}$

Variable selection in linear regression

model: $\hat{y} = \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$ for $n = 1, 2, \ldots, 20$ and $N = 50$



- aim to choose the number of basis function (n)
- set the effective number of parameters $d = 2n$ (the number of $\sin(kx), \cos(kx)$)
- compute ΔAIC, ΔAICc, ΔBIC (subtracted by its minimum), C_p, adjusted R^2

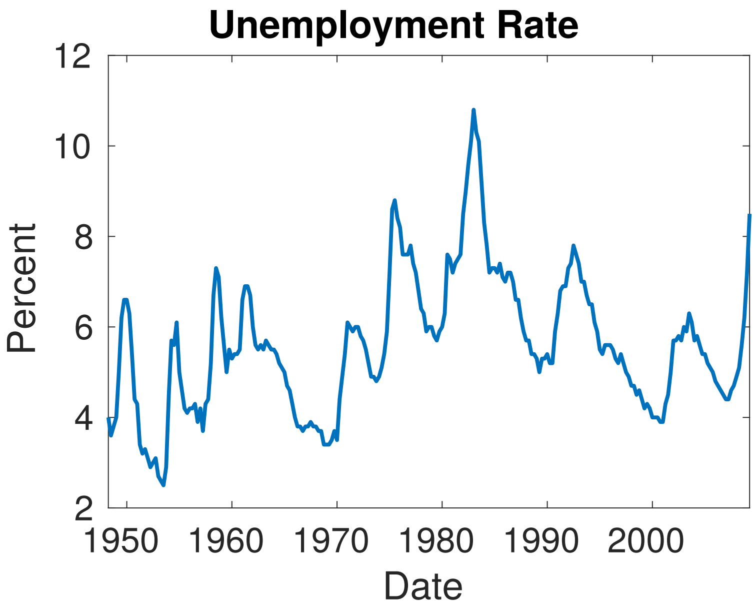


- AIC and adjusted R^2 chose a complex model, while AICc and BIC picked 4 basis functions (simpler), and C_p chose 7 basis functions

- train MSE always decreases, as well as, R^2 always increases but the curves have a knee around $n = 4$

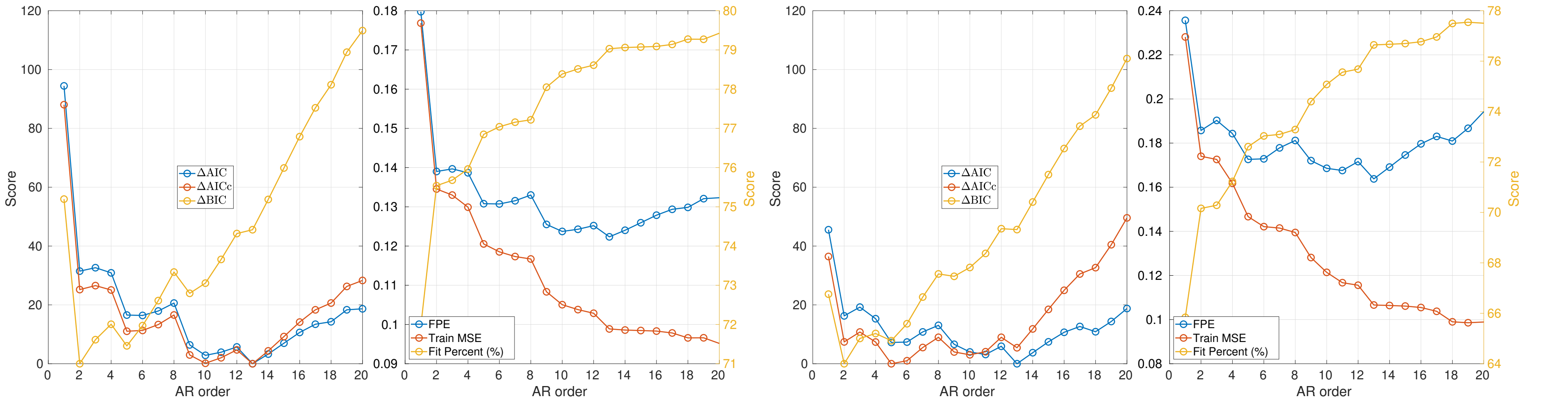
Choosing AR lag order

fitting AR model of order $p = 1, 2, \ldots, 20$ to unemployment rate time series



- the effective number of parameters is chosen as $d = p$
- compute ΔAIC, ΔAICc, ΔBIC, FPE, train MSE, and Fit Percent
- data samples: $N = 245$, examine two cases: (i) use all data (ii) use only half

left: use all data right: use half of data



- left: AIC, AICc and FPE tend to choose a higher order model $(p = 13)$ but BIC prefers a simpler model $(p = 2)$

- right: AICc chose a lower order model when N is halved (sample size was corrected)

- both *train* MSE and Fit Percent are not good indicators for model selection

Log-likelihood based scores (AIC, AICc)

AIC, AICc, BIC use negative log-likelihood to indicate model quality

$$\begin{aligned}
\text{AIC}(\alpha) &= -2\mathcal{L}(\alpha) + 2d \\
\text{AICc}(\alpha) &= -2\mathcal{L}(\alpha) + 2d + \frac{2d(d+1)}{N-d-1} \\
\text{BIC}(\alpha) &= -2\mathcal{L}(\alpha) + d\log N
\end{aligned}$$

- AIC is an approximation of Kullback-Leibler (KL) divergence between the true density $(f(x)$ and the model ($g(x|\hat{\theta})$)

$$\begin{aligned}
I(f,g) &= \int f(x)\log(f(x)/g(x|\theta))dx \\
-\mathcal{L}(\hat{\theta}) + d &\approx \mathbf{E}_{\hat{\theta}}[I(f(x), g(x|\hat{\theta}))] + \text{constant}
\end{aligned}$$

- AICc penalizes more on complexity for small N (as quadratic term in d); it approaches AIC for large samples (large N)

Log-likelihood based score (BIC)

- BIC penalizes more on complexity than AIC (as indicated by $\log N > 2$)
- when model candidates contain a true model, BIC is consistent (probability of choosing the correct model $\to 1$ as $N \to \infty$)
- model with minimum BIC $\Leftrightarrow$ model with *highest* posterior density

$$\text{posterior odds} = \frac{P(\mathcal{M}_m|\text{data})}{P(\mathcal{M}_l|\text{data})} = \underbrace{\frac{P(\mathcal{M}_m)}{P(\mathcal{M}_l)}}_{\text{prior}} \cdot \underbrace{\frac{P(\text{data}|\mathcal{M}_m)}{P(\text{data}|\mathcal{M}_l)}}_{\text{Bayes factor}}$$

model prior tells which model is more likely to be preferred (by users)

- when prior is not available (all models have equal probabilities), Bayes factor directly affects the posterior odds
- BIC (with -2 factor) is an approximate of Bayes factor (see Hastie et al. book)

- for nested models $\mathcal{M}_1$ (complex), $\mathcal{M}_2$ (simple) with $d(\mathcal{M}_1) = d(\mathcal{M}_2) + m$
  - AIC picks complex model if $\mathcal{L}(\mathcal{M}_1) - \mathcal{L}(\mathcal{M}_2) > 2m$ (it's worth to use complex model since model quality improved much more)
  - BIC picks complex model if $\mathcal{L}(\mathcal{M}_1) - \mathcal{L}(\mathcal{M}_2) > m \log N$

- improved gap of log-likelihood required by AIC is less than that of BIC; hence, AIC is prone to choosing a complex model more easily than BIC

- for LR (log-likelihood ratio) test, with test statistic

$$2(\mathcal{L}(\mathcal{M}_1) - \mathcal{L}(\mathcal{M}_2)) \sim \mathcal{X}^2(m)$$

  - LR test picks $\mathcal{M}_1$ (complex) if $2\mathcal{L}(\mathcal{M}_1) > 2\mathcal{L}(\mathcal{M}_2)$ by $\mathcal{X}^2_{0.05}(m)$
  - for $m < 7$, we have $2m < \mathcal{X}^2_{0.05}(m)$; hence, AIC tends to pick a complex model more easily than LR test in this case

Akaike's final prediction (FPE)

denote $V(\hat{\theta})$ a loss function used in prediction error method (*e.g.*, det or trace of error covariance)

$$\text{FPE}(\alpha) = V(\hat{\theta}) \left(\frac{1 + d/N}{1 - d/N} \right)$$

- model complexity is cooperated in *multiplicative form* (as compared to additive form in AIC, BIC)
- when model output is scalar, $V(\hat{\theta})$ is simply MSE and FPE reduces to

$$\text{FPE} = \frac{1}{N} \sum_{t=1} \varepsilon^2(t, \hat{\theta}) \cdot \frac{1 + d/N}{1 - d/N}$$

- it was shown in Ljung book that FPE is a way to approximate of $\lim_{N \to \infty} \mathbf{E}[V(\theta)]$ (population), which can be estimated using $V(\hat{\theta})$ evaluated on *estimation data*

Mallow's C_p

C_p is mostly used in linear regression with d predictors and homoskedastic noise

$$C_p(\alpha) = \frac{1}{N}\left[\mathrm{RSS}(\alpha) + 2d\hat{\sigma}^2\right]$$

- C_p uses *quadratic loss* to measure model quality
- $\hat{\sigma}^2$ is an estimate of noise variance using **full** model
- RSS$/N$ always decreases when d increases; penalty on complexity is put on $2d\hat{\sigma}^2$
- in Hastie et al. book, it showed that C_p is an estimate of test MSE
- other form of C_p exists: $C_p = \mathrm{RSS}/\hat{\sigma}^2 + 2d - N$ but result in choosing the same d

Adjusted R^2

R^2 (coefficient of determination) is based on the decomposition:

$$\underbrace{\sum_i (y_i - \bar{y})^2}_{\text{TSS}} = \underbrace{\sum_i (y_i - \hat{y}_i)^2}_{\text{RSS}} + \underbrace{\sum_i (\hat{y}_i - \bar{y})^2}_{\text{ESS}} + \underbrace{2\sum_i (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})}_{\text{zero if model has a constant}}$$

R^2 is the proportion of the total variation in Y that can be linearly predicted by X

$$R^2 = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}}, \quad \text{adjusted } R^2 = 1 - \frac{\mathrm{RSS}(\alpha)/(N-d-1)}{\mathrm{TSS}/(N-1)}$$

- for linear model, $0 \leq R^2 \leq 1$ and always increases for larger models
- the presence of d penalizes the criterion for the number of predictor variables
- adjusted R^2 increases if the added predictor variables decrease RSS enough to compensate for the increase in d

Score relations

for Gaussian noise additive model, we can show that log-likelihood (up to constant) is

$$-2\mathcal{L}(\theta) = \begin{cases} N \log\det\left(\frac{1}{N}\sum_{t=1}\varepsilon(t,\theta)\varepsilon(t,\theta)^T\right), & \text{if noise covariance is a parameter} \\ \frac{\mathrm{RSS}(\theta)}{\sigma^2}, & \text{if noise variance is given} \end{cases}$$

- for scalar output and noise variance is a parameter

$$\mathsf{AIC}_{\mathsf{scaled}} = \mathsf{AIC}/N = \log(\mathsf{MSE}) + 2d/N \approx \log(\mathsf{FPE}), \quad \text{for } d \ll N$$

- for scalar output and noise variance is given as $\hat{\sigma}^2$ (as computed from full model)

$$\mathsf{AIC}_{\mathsf{scaled}} = \mathsf{AIC}/N = \frac{1}{N}\left(\frac{\mathrm{RSS}(\theta)}{\hat{\sigma}^2} + 2d\right) = \hat{\sigma}^2 \cdot C_p$$

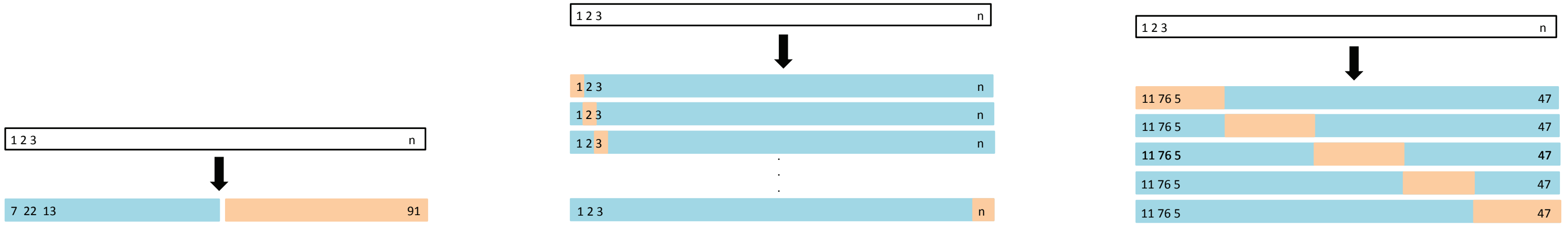
AIC and C_p choose the same model in this case

Cross validation

- training error rate: the average error that results from using a trained model (or method) back on the training data set
- test error rate: the average error that results from using a statistical learning method to predict the response on a **new observation**
- training error can be quite different from the test error rate
- **cross validation** can be used to estimate *test error rate* using available data: split into training and validation sets
  - validation set approach
  - leave-one-out cross validation
  - k-fold cross validation

Splitting data

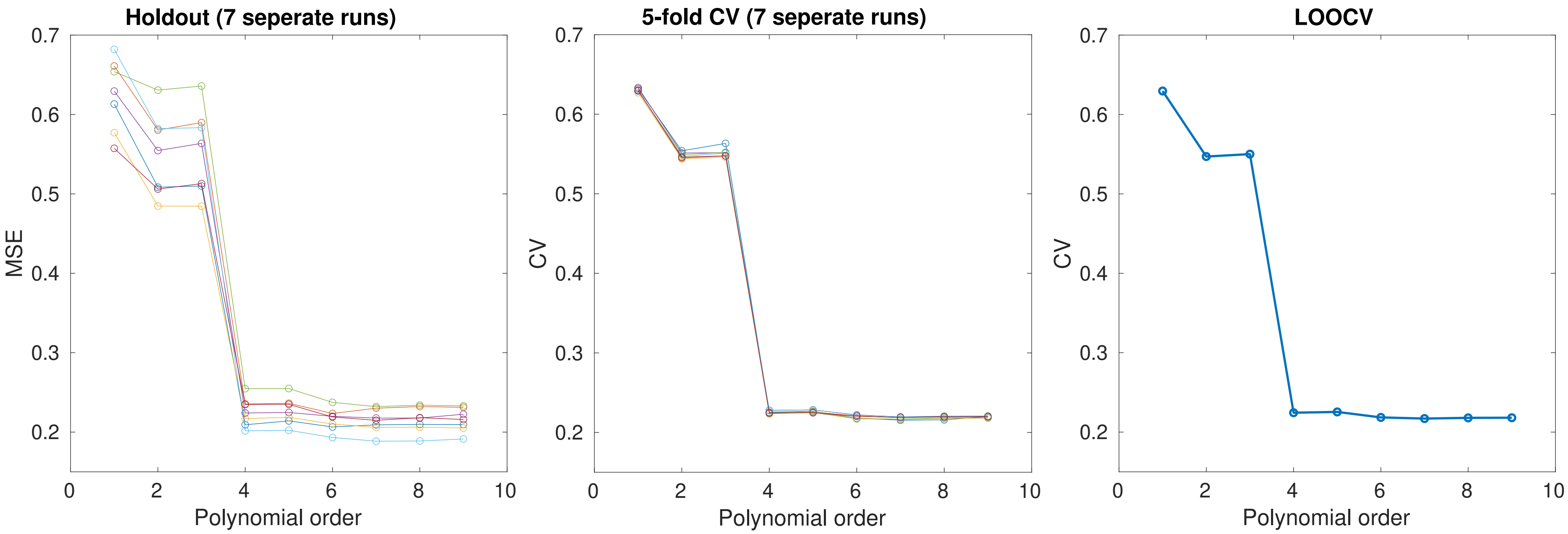
- training set: used for fitting a model
- validation set: used for predicting the response from the fitted model



- validation set approach or hold out (left): randomly split data
- leave-one-out or LOOCV (middle): leave 1 sample for validation set
- k-fold (right): randomly split data into k folds; leave 1 fold for validation
  - repeat k times where each time a different fold is regarded as validation set and compute $\text{MSE}_1, \text{MSE}_2, \ldots, \text{MSE}_k$
  - the test error rate is estimated by **averaging** the k MSE's

Cross validation on polynomial order

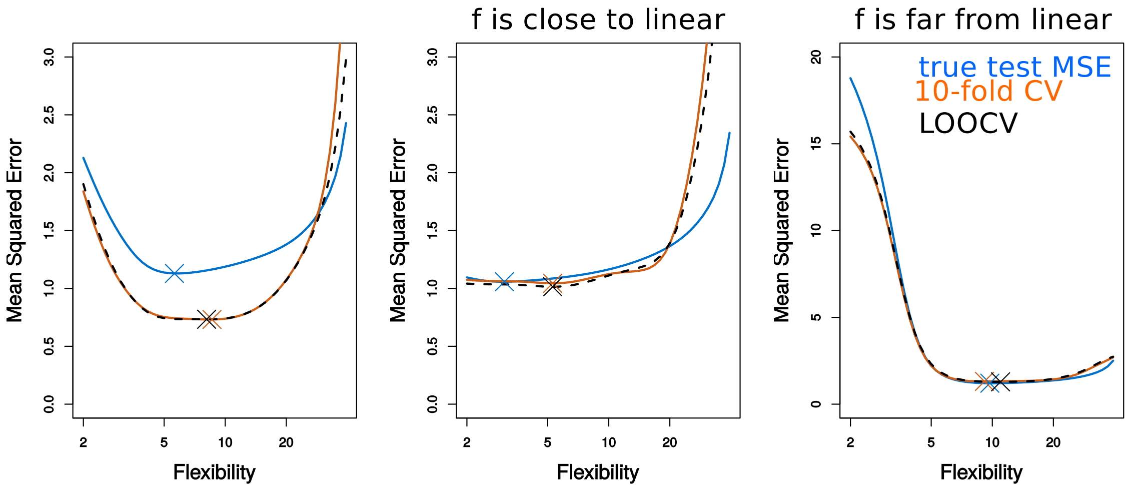
$N = 500$, show 7 runs of holdout, and 5-fold



- result of holdout has high variation since it depends on random splitting
- 5-fold results has less variation because MSE is averaged over k folds
- LOOCV requires N loops (high computation cost); MSE$_i$'s are highly correlated

Estimate a true test MSE by CV

accuracy of test error rate (on simulation data set): using model of smoothing splines

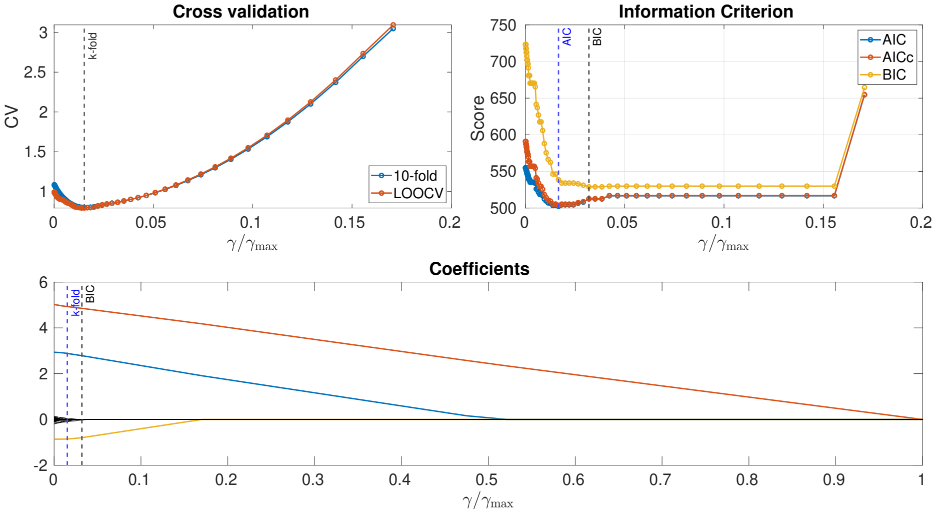


compute the *true test MSE* (assume to know true f) as a function of complexity

- (left): cv estimates have the correct general U shape but underestimate test MSE
- (center): cv gives overestimate of test MSE at high flexibility
- (right): the true test MSE and the cv estimates are almost identical

Choosing penalty parameter in lasso

lasso can be used for feature selection by choosing a right amount of penalty



- each penalty parameter γ corresponds to a sparsity pattern of β
- vary γ and evaluate model selection scores and CV
- k-fold, LOOCV, AIC and AICc chose smaller γ than the one selected by BIC
- solution path show the significant β_i's selected from all methods

Model validation

the parameter estimation procedure picks out the *best* model

a problem of model validation is to verify whether *this best* model is “good enough”

general aspects of model validation

- validation with respect to the purpose of the modeling
- feasibility of physical parameters
- consistency of model input-output behavior
- model order reduction
- parameter confidence intervals
- simulation and prediction

Validation of dynamical models

dgp: $y = Gu + He$

model: $\hat{y} = \tilde{G}u + \tilde{H}e$

residual error: $\varepsilon(t) = y(t) - \hat{y}(t)$

common validation approaches based on residual analysis

- whiteness test of residuals
- cross-correlation test (between residual and input)
- examination of model order

Whiteness test of residuals

residual error contain mismatch in G (system dynamic) and H (noise dynamic)

$$\varepsilon(t) = (G - \hat{G})u + (H - \hat{H})e, \qquad R_\varepsilon(\tau) = \frac{1}{N}\sum_{t=\tau}^{N} \varepsilon(t)\varepsilon(t-\tau)$$

- $\varepsilon(t)$ can be regarded as filtered noise if there is a model mismatch in H and $R_\varepsilon(\tau)$ is not significantly small at $\tau \neq 0$ ($y(t)$ could have been better predicted)
- apply hypothesis test (H_0: ε is white) with test statistic

$$W = \frac{N}{R_\varepsilon^2(0)} \sum_{\tau=1}^{m} R_\varepsilon^2(\tau) \xrightarrow{d} \chi^2(m)$$

- if $W > \mathcal{X}_\alpha^2(m)$, we reject H_0 (reject the model and improve $\hat{H}$)

Cross-correlation test

if $\hat{G}$ perfectly matches G, residual ε contains no dynamic of u, so the cross-correlation

$$R_{\varepsilon u}(\tau) = \frac{1}{N}\sum_{t=\tau}^{N} \varepsilon(t)u(t-\tau)$$

must be zero for all τ

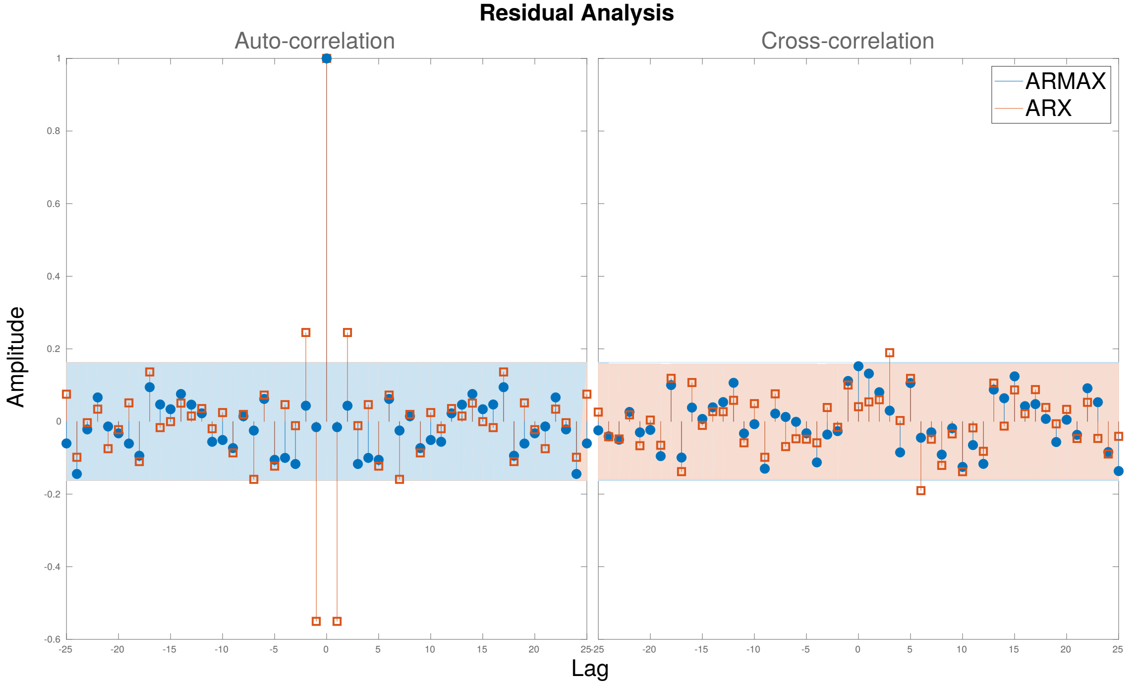
- form a hypothesis test with $H_0 : R_{\varepsilon u}(\tau)$ is zero
- we can compute the test statistic

$$W = Nr^T[R_{\varepsilon}(0)R_u]^{-1}r \xrightarrow{d} \chi^2(m)$$

  r is a sample cross-correlation, R_ε and R_u are auto-correlation
- if $W > \mathcal{X}^2_\alpha(m)$, we reject H_0 (reject the model and improve $\hat{G}$)

Residual analysis of ARMAX model

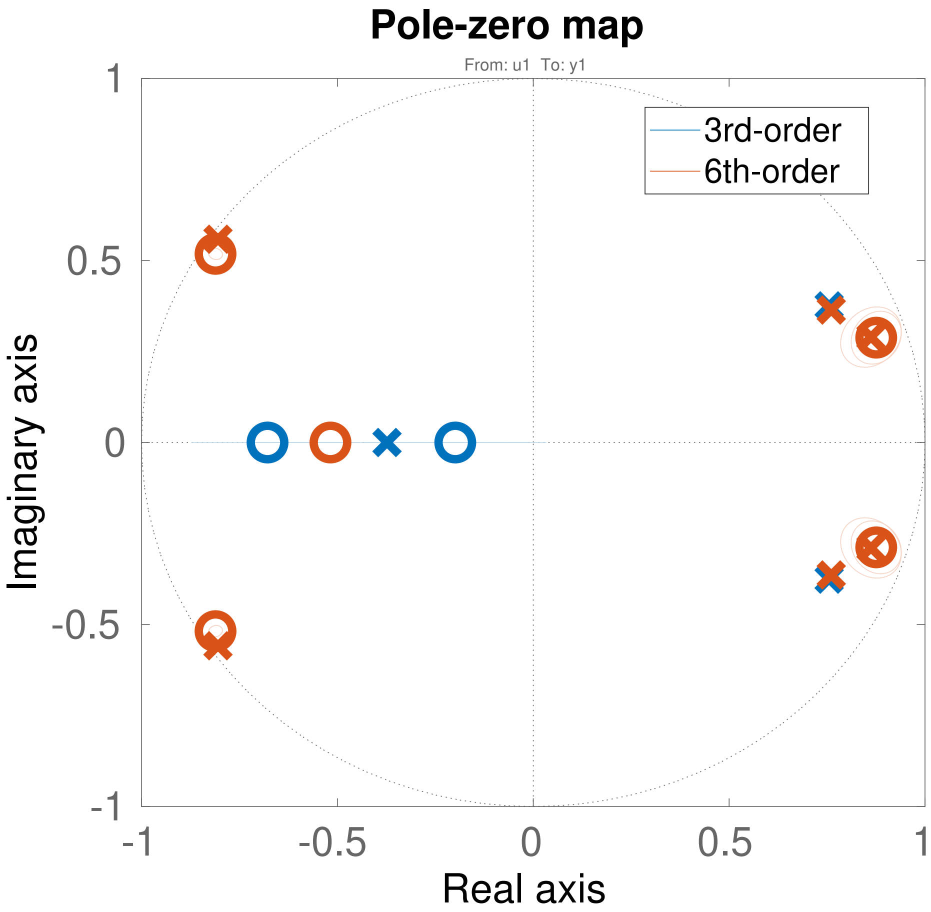
true system: ARMAX(2,2,3) and consider models ARX(3,3) and ARMAX(3,3,3)



- ARX has a significant $R_\varepsilon(3)$ (more apparent than ARMAX) — because ARX does not incorporate noise dynamic in the model
- $R_{\varepsilon u}$ of both model stay inside the acceptable region ($\hat{G}$ was suitably estimated)

Model order examination

if a model is *overparametrized*, it is more likely to see zero-pole cancellation



compare ARMAX models of order (3,3,3) and (6,6,6)

Example of MATLAB commands

- `resid`: residual analysis
- `compare`: compare the prediction with the measurement
- `iopzplot`: plots of zeros and poles

References

- T. Hastie, R. Tibshirani, and J. Friedman, *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, Second edition, Springer, 2009

- G.James, D. Witten, T. Hastie, and R. Tibshirani, *An Introduction to Statistical Learning*, Springer, 2013

- T. Söderström and P. Stoica, *Chapter 11: System Identification*, Prentice Hall, 1989

- L. Ljung, *Chapter 16: System Identification: Theory for the User*, 2nd edition, Prentice Hall, 1999

- K.P. Burnham and D.R. Anderson, *Model selection and multimodel inference: a practical information-theoretic approach*, Springer, 2002