Estimation of Autoregressive with Exogeneous Inputs Model for fMRI Data

Nattaporn Plub-in, Patawee Prakrankamanant, Nop Polboon
Ranyaphat Hongpipatsak, Morokot Cheat, Jitkomut Songsiri
Department of Electrical Engineering
Faculty of Engineering
Chulalongkorn University

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Contents

Abstract

Functional Magnetic Resonance Imaging (fMRI) is one of the most popular methods to measure human brain activities. The fMRI based on BOLD effect describes the activated brain region according to neuron activity in order to analyze brain connectivity. This study aims to estimate the BOLD response using an autoegressive model with exogenous input (ARX). A least-square technique is applied to fit model. Since the AR coefficients have a joint zero pattern due to Granger causality, the results were generated by two models: ARX with unconstraint and ARX with zero constraint. The results of this study presented the performance of two models using the differrence between true data and estimated data. Finally, the accuracy of two models was discussed.

1 Introduction

Brain connectivity is one of the most popular topics of neuroscience. It is extremely challenging to explore how regions in brain communicate with each other since there no model truly explaining about brain connectivity. Nowadays, brain activity can be interpretted in different ways with different methods such as electroencephalogram (EEG), magnetoencephalogram (MEG) and functional magnetic resonance imaging (fMRI). This project estimated brain connectivity by using fMRI widely utilized for measuring brain function activity. fMRI is modified from magnetic resonance imaging (MRI) [8] by using neuroimaging technology to generate 3D images of brain. In addition, it is also comfortable to be geometrically interpretted in term of 2D data by concerning the Blood-oxygenation-level-dependent (BOLD) signals of each voxel.

This paper studies on brain connectivity estimation using ARX model studying from the previous paper [11] using AR model and concerning Granger causality which is a statistical concept of causality that is based on prediction. According to this concept, if a signal y_i is Granger-caused by a signal y_j, then past values of y_j helps predict furure values of y_i more than only using past values of y_i.

This paper developes from the estimation brain connectivity in [11] stating that brain connectivity has Granger causality properties. If the states at time series y_i is Granger-caused by time series y_j, the past values of y_j will help to improve the prediction of y_i. [11] had estimated brain connectivity by autoregressive model with Granger causality properties as (1).

$$y(t) = A_1 y(t-1) + A_2 y(t-2) + \cdots + A_p y(t-p) + u(t) \tag{1}$$

where $y(\cdot) \in \mathbf{R}^n$, $A_k \in \mathbf{R}^{nxn}$, $k = 1, 2, \ldots, p$ and $u(\cdot)$ is input noise.
(we denote $(A_k)_{ij}$ the (i, j) entry of matrix A_k.)
In this project, we add input variables into AR model from (1) then the model turn into auto regressive model with exogenous input (ARX) as (2).

$$y(t) = \sum_{k=1}^{p} A_k y(t-k) + \sum_{k=1}^{q} B_k u(t-k) + e(t) \tag{2}$$

where y is the observed BOLD signal, u is controlled variables such as change in stimuli (visual cue or auditory signal) and e is noise.

We use ARX model because model with input variables can be estimated the brain connectivity more precise by input variables are describe in section 3. A main goal of our project is analysis and simulations of ARX model can describe the brain connectivity with more precise and accuracy. The following section describes details of fMRI, the type of signals that we got from fMRI and the basic concept of Granger causality. In section 3, we present overview of ARX model and estimation method including the description of data which we use in this project. In section 4, we describe experiment and result of this work. Lastly, the conclusion will be discusses in section 5.

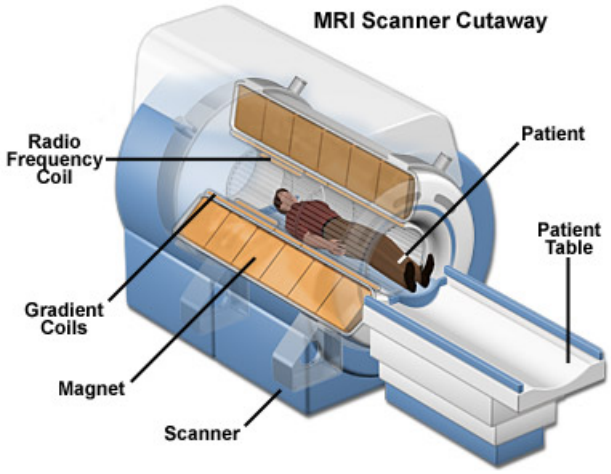


Figure 1: MRI Scanner
source :
http://www.clipmass.com/story/35886

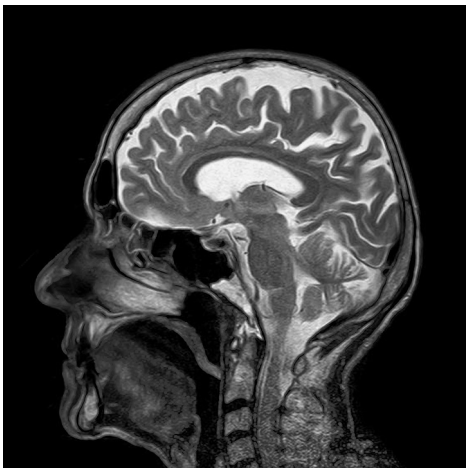


Figure 2: MR image
source : https://pixabay.com/

2 fMRI

2.1 Acquiring of fMRI data

Magnetic resonance imaging (MRI) is a medical imaging technique used in radiology to form 3D images of the anatomy and the physiological processes of the body using MRI machine as shown in Figure 1. The brief principle is that MR scanner release radio wave to align Hydrogen atoms existing within an interesting regions. When the energy are emitted back for returning to the resting state of atoms, the MR scanner captures this energy by the coils. Next, the computer processes the signals and generates a series of 2D grey-scaled images from the 3D image. As shown in Figure 2 is the example of MR image.

Functional magnetic resonance imaging (fMRI) modified from MRI is a medical imaging technique which maps patterns of brain activation with high spatial resolution by measuring changes in blood. The fMRI bases on the BOLD effect consisting of the two fact. The first fact is that when hemoglobin, the cell in blood that carries O_2, loses O_2 to become deoxyhemoglobin, the magnetic properties change. The sencond fact is that when an area of brain is activated, the blood flow increases. These produce the BOLD effect, a local increase of the MR signal owing to a reduction of the oxygen extraction fraction during increased neural activity. An fMRI image shows the result in term of 3D image. The 3D image is a volume resolution element as called voxel. Considering each voxel for all sampling time, the plot between the intensity of image in the voxel and time is called a BOLD signal shown in Figure 3.

The basis of most of the fMRI studies of brain activation performed today is BOLD signals based the fact that fully oxygenated blood has about the same susceptibility as other brain tissues, but deoxyhemoglobin is paramagnetic and changes the susceptibility of the blood. When regions become more deoxygenated, field distortions around the vessels are increased by MRI machine detection, and the BOLD signal decreases. On the contrary, if blood oxygenation increases, the BOLD signal also increases. That means deoxyhemoglobin has the effect of restraint the BOLD signal, while oxyhemoglobin does not [8]. As shown in 4, the left figure illustrates a resting region which has a small blood flow of both deoxyhemoglobin and oxyhemoglobin. Because of a small difference of blood susceptibility, BOLD signals are not affected much. In the contrary, the right figure illustrates a activated region. When a region is activated, a lot of oxygenated blood flow raises and then drops by becoming deoxygenated blood. Thus, when the region is activated, BOLD

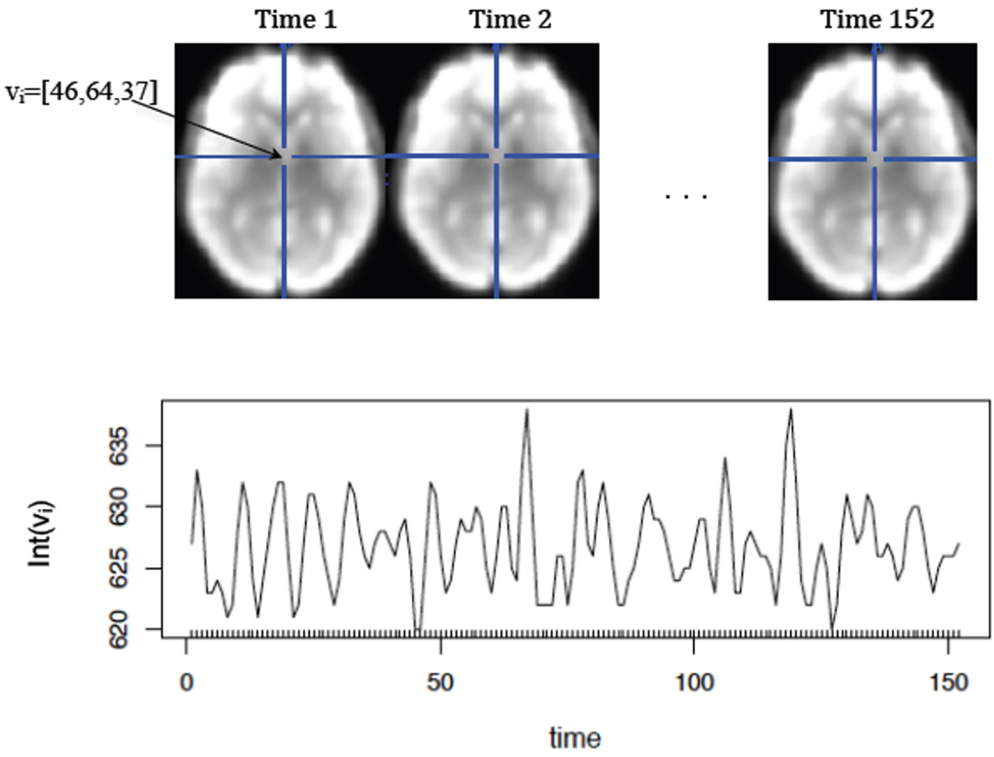


Figure 3: A plot between the intensity of image at voxel v_i and time points [4]

signals raises and drops [3]. Then, BOLD signals of each voxel, a volumn resolution element in 3D image, are recorded as 2D graph expressing time-series of BOLD signals

BOLD signals are not obtained directly from neuronal activities but via hemodynamic response function (HRF). Figure 5 illustrates the model how we get the BOLD signals from neuronal activities via HDR and fMRI BOLD response expressed in a nonlinear system. To manage the problem of a nonlinear system, the Balloon model is sufficient to account for nonlinear behaviors [5].

The input-output behavior of this model can be compared to the real brain in terms of their respective Volterra kernels.

2.2 Overview of ARX model

In this study, the estimation of brain connectivity uses autoregressive model with exogenous input (ARX). Because AR processes are traditional approach to statistical characterization of the time series [10]. The exogenous input will be utilized if and only if it help to improve the prediction. ARX model is more suitable than AR when we have the high demension such as fMRI data. The ARX model based on a linear model which be expressed (2). From (2), we will estimate matrix $A = [A_1 \;\; A_2 \ldots A_p]$ and $B = [B_1 \;\; B_2 \ldots B_q]$. The matrix A describes the relation between voxels and matrix B describes the relation between an external input and each voxel.

The least-square (LS) is the general approach that be produced to determine the best fit line to data. In this study, we will employ a least-square method for fitting ARX model to the measurements $y(1), y(2), \ldots, y(N)$. The model parameters (A_k and B_k) are chosen such that the quadratic loss $\sum_{t=p+1}^{N} \|y_t - \sum_{k=1}^{p} A_k y(t-k) - \sum_{k=1}^{q} B_k u(t-k)\|^2$ is minimized. If we define $A \in \mathbf{R}^{n \times np}$ and $B \in \mathbf{R}^{n \times mq}$. So, we assume $\theta = [A \;\; B]$ and can be rewritten more compactly as $\|Y - \theta H\|_2^2$ where

$$Y = \begin{bmatrix} y(p+1) & y(p+2) & \ldots & y(N) \end{bmatrix}$$

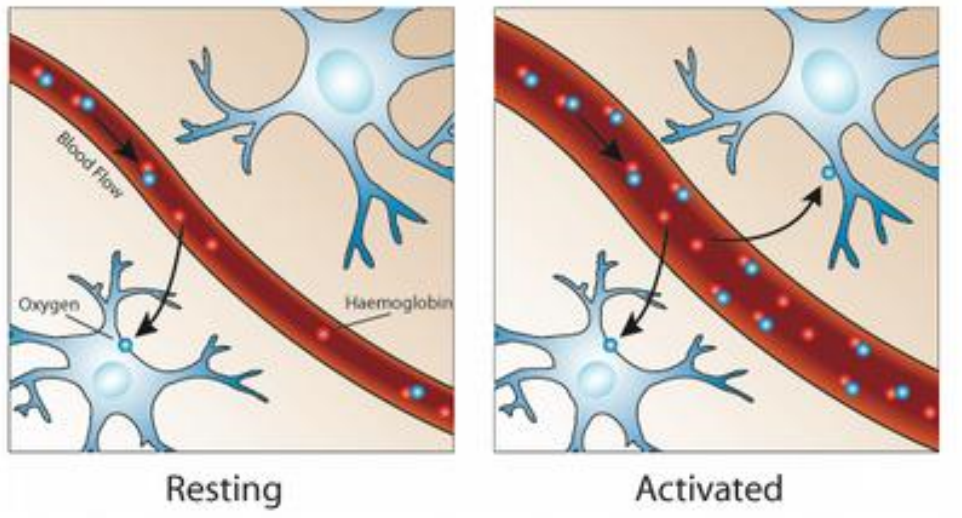


Figure 4: Illustration of the BOLD mechanism. During activated periods, the blood flow increases and more oxygenated than resting then, the location of neronal activies are found.
source : `http://www.sbirc.ed.ac.uk/research/techniques/functional.html`

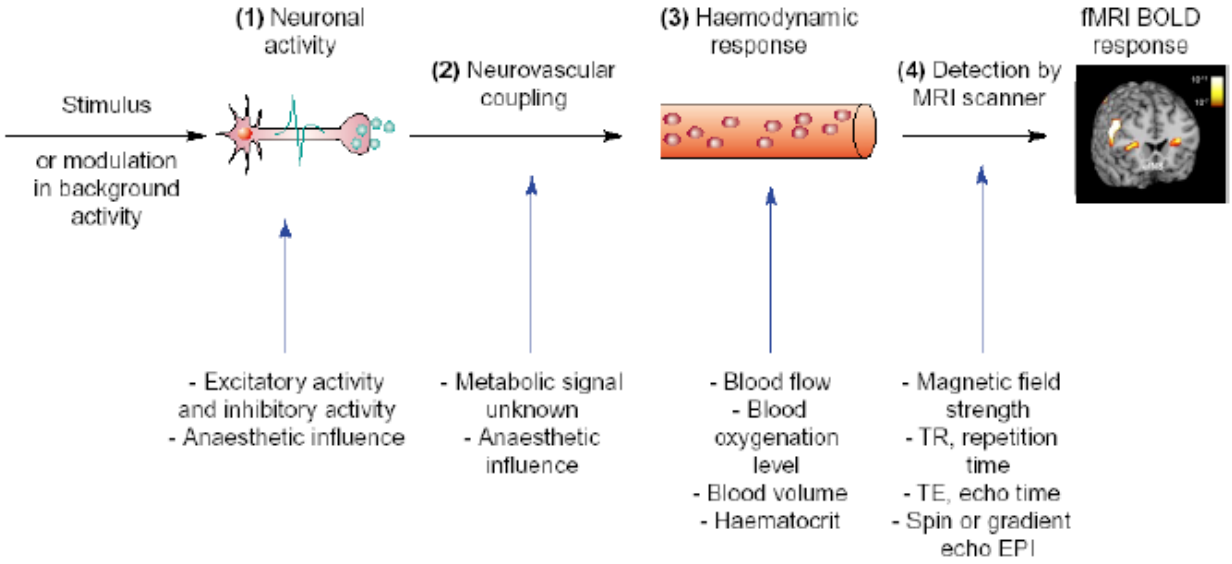


Figure 5: The BOLD signal has several constituents: (1) the neuronal response to a stimulus or background modulation; (2) the complex relationship between neuronal activity and triggering a haemodynamic response (termed neurovascular coupling); (3) the haemodynamic response itself; and (4) the way in which this response is detected by an MRI scanner [1].

$$H = \begin{bmatrix} y(p) & \cdots & y(N-1) \\ \vdots & \ddots & \vdots \\ y(1) & \cdots & y(N-p) \\ u(p) & \cdots & u(N-1) \\ \vdots & \ddots & \vdots \\ u(p-q) & \cdots & u(N-q) \end{bmatrix}$$

$$Y = \theta H + e \tag{3}$$

Thus, the least-square estimation of unconstraints and zero constraints which based on the Granger causality is applied to predict the coefficient of matrix. For this technique, we minimize the error between data and model's outputs and the coefficient of matrix θ will be estimated for fitting the data, then the error of (2) will be minimized by following the sentence below

$$\underset{\theta}{\text{minimize}} \quad \|Y - \theta H\|_F^2$$

The optimal solution is found from setting derivative to zero

$$\frac{d}{d\theta} \|Y - \theta H\|_F^2 = 0 \tag{4}$$

Finally, we found a closed-form solution of the least square estimation of unconstrained ARX as shown in (5). This solution will exist if and only if, H is full rank.

$$\theta = YH^T(HH^T)^{-1} \tag{5}$$

2.3 Granger Causality (GC)

The basic concept of Granger Causality (GC) is straightforward. Suppose that we have two states y_i and y_j. If the states at time series y_i is Granger-caused by time series y_j, the past values of y_j will help improve the prediction of y_i. The computation of GC is suitable for autoregressive processes [2]. Referring to [6], they applied GC to the explanation of interregional connectivity in fMRI data and to detection of the direction of information flow between brain regions. There experiment considered a k -dimension vector autoregressive model (VAR) of order p. In the part of results, they got the important results of VAR model that are the series y_j non-causes y_i, if and only if, the coefficient $(A_k)_{ij} = 0$ for any i. Hence, the zero pattern of A_k is necessary to consider and will be applied to analyze time series of BOLD signals in fMRI data.

1) Pairwise Granger Causality

Pairwise is a popular approach for testing fMRI data and is simply to following the solution [2]. Assuming $x(t)$ and $y(t)$ are stochastic and Wide-Sense stationary (WSS). The form of AR models of those processes is followed the equation as

$$x(t) = \sum_{k=1}^{p} a_{1k} x(t-k) + \epsilon_1(t), \quad \text{var}(\epsilon_1(t)) = \Sigma_1$$

$$y(t) = \sum_{k=1}^{p} c_{1k} y(t-k) + \nu_1(t), \quad \text{var}(\nu_1(t)) = \Sigma_1^{'}$$

If $x(t)$ and $y(t)$ are a relative cause, we can rewrite the equation again as

$$x(t) = \sum_{k=1}^{p} a_{2k} x(t-k) + \sum_{k=1}^{p} c_{2k}^{'} y(t-k) + \epsilon_2(t) \tag{6}$$

$$y(t) = \sum_{k=1}^{p} c_{2k} y(t-k) + \sum_{k=1}^{p} a_{2k}^{'} x(t-k) + \nu_2(t) \tag{7}$$

Then, we define $q(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \sum_{k=1}^{p} H_k q(t-k) + \gamma(t)$ and the covariance matrix of noise is var $(\gamma(t)) = \begin{bmatrix} \Sigma_2 & \mathcal{T} \\ \mathcal{T}^T & \Sigma_2^{'} \end{bmatrix}$

i.e., $\text{var}(\epsilon_2(t)) = \Sigma_2$, $\text{var}(\nu_2(t)) = \Sigma_2^{'}$ and $\mathcal{T} = \text{cov}(\epsilon_2(t), \nu_2(t))$
If $x(t)$ and $y(t)$ are independent, c_2 and $c_2^{'}$ are zero. We can say that $\mathcal{T}$ is zero.

From [2], we can find the quantity of $y(t)$ which is a Granger-cause on $x(t)$ as

$$F_{y \to x} = \ln \frac{\Sigma_1}{\Sigma_2}$$

If $F_{y \to x}$ is more than zero, $y(t)$ will be cause on $x(t)$. But $F_{y \to x}$ equals to zero, $y(t)$ is non-causes $x(t)$. In practice, pairwise GC is not good for multivariate data [2] because it has a limitation of relative description. The Figure 6 displays the relative cause of $y(t)$ and $z(t)$ have effect to $x(t)$ and the Figure 7 shows $y(t)$ has effect to $x(t)$ via $z(t)$. In this event, the relation is not correct. Hence, the conditional GC will be used to find the cause between the variables when we have multivariate data.

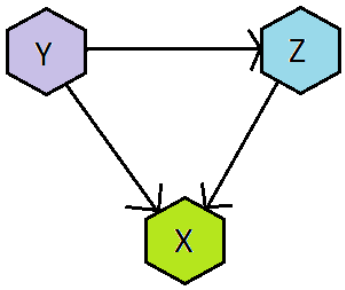


Figure 6: $y(t)$ has both directly and indirectly to $x(t)$

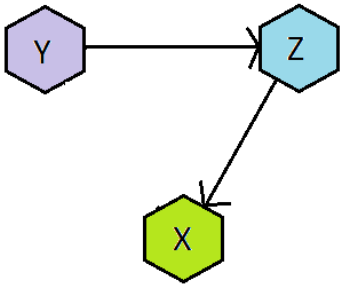


Figure 7: $y(t)$ has indirectly affect to $x(t)$

2) Conditional Granger Causality

This method can manage the GC of the multivatiate data. Assuming $x(t)$, $y(t)$ and $z(t)$ are WSS. To find $y(t)$ is cause on $x(t)$ with $z(t)$ condition or not. The equation is following.

$$x(t) = \sum_{k=1}^{p} a_{3k} x(t-k) + \sum_{k=1}^{p} e_{3k}^{'} z(t-k) + \epsilon_3, \quad \text{var}(\epsilon_3) = \Sigma_3$$

$$z(t) = \sum_{k=1}^{p} e_{3k}^{'} z(t-k) + \sum_{k=1}^{p} a_{3k} x(t-k) + \nu_3, \quad \text{var}(\nu_3) = \Sigma_3^{'}$$

Then, the covariance matrix of noise between $x(t)$ and $z(t)$ are defined as $\Sigma = \begin{bmatrix} \Sigma_3 & \mathcal{T} \\ \mathcal{T}^T & \Sigma_3^{'} \end{bmatrix}$

If $x(t)$, $y(t)$ and $z(t)$ are a relative cause, we can rewrite the equation again as

$$x(t) = \sum_{k=1}^{p} a_{4k} x(t-k) + \sum_{k=1}^{p} c_{4k}^{'} y(t-k) + \sum_{k=1}^{p} e_{4k}^{'} z(t-k) + \epsilon_4 \tag{8}$$

$$y(t) = \sum_{k=1}^{p} a_{4k}^{*} x(t-k) + \sum_{k=1}^{p} c_{4k} y(t-k) + \sum_{k=1}^{p} e_{4k}^{*} z(t-k) + \nu_4 \tag{9}$$

$$z(t) = \sum_{k=1}^{p} a_{4k}^{'} x(t-k) + \sum_{k=1}^{p} c_{4k}^{*} y(t-k) + \sum_{k=1}^{p} e_{4k} z(t-k) + \lambda_4 \tag{10}$$

We define the covariance matrix of noise as $\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{bmatrix}$ Look like the pairwise method to find the quantity of $y(t)$ is cause on $x(t)$ with $z(t)$ condition as shown below :

$$F_{y \to x|z} = \ln \frac{\Sigma_3}{\Sigma_{xx}}$$

We found that if $F_{y \to x|z}$ is more than zero, $y(t)$ will be cause on $x(t)$ with $z(t)$ condition. In other word, if $\Sigma_3 = \Sigma_{xx}$, $F_{y \to x|z}$ equals to zero. So, $y(t)$ is not cause on $x(t)$ with $z(t)$ condition.

Both of pairwise and conditional GC can lead to the zero pattern of A_k. If $y(t)$ is not direct cause on $x(t)$, the coefficient will be zero. *i.e.*, the some element of $(A_k)_{ij}$ is zero. For fMRI data sets, the conditional GC is more appropriate than pairwise GC since we can set the other voxels that are not interest to be the conditional terms.

3 Estimating formulation

In this section, we present the formulation for finding the brian connectivity. *i.e.*, A and B will be estimated using ARX both with and without constraints. The ARX with zero constraints will be explained as a straigthforward way. After that, the details of data neuroimage(2013) from [7] will be described how we get BOLD signals.

3.1 ARX model with zero constraints

If the Granger Causality is given, the problem of estimating ARX model subjects to zero constraint (i.e a brain network topology is known and our data contains the zero pattern of A_k).

Next, we will define the new term of matrix below

$Y = [y_1 \ y_2 \dots y_N]$ where $Y \in \mathbf{R}^{n \times N}$

$A = [A_1 \ A_2 \dots A_p]$ where $A \in \mathbf{R}^{n \times np}$ and $A_k \in \mathbf{R}^{n \times n}$ for $k = 1, 2, \dots, p$

$B = [B_1 \ B_2 \dots B_q]$ where $B \in \mathbf{R}^{n \times mq}$ and $B_k \in \mathbf{R}^{n \times m}$ for $k = 1, 2, \dots, q$

where n is a number of voxel in brain, N is a number of measurement and p is the index of Granger causality data (the position that we know the data is Granger).

By then, we define

$$H = \begin{bmatrix} y(p) & \cdots & y(N-1) \\ \vdots & \ddots & \vdots \\ y(1) & \cdots & y(N-p) \end{bmatrix} = \begin{bmatrix} H_1 \\ \vdots \\ H_p \end{bmatrix} \text{ and } K = \begin{bmatrix} u(p) & \cdots & u(N-1) \\ \vdots & \ddots & \vdots \\ u(p-q) & \cdots & (N-q) \end{bmatrix} = \begin{bmatrix} K_1 \\ \vdots \\ K_q \end{bmatrix}$$

From (2) above, we can minimize the error between data and model's outputs below

$$\begin{aligned} \underset{A,B}{\text{minimize}} \quad & \|Y - AH - BK\|_F^2 \\ \text{subject to} \quad & (A_1)_{ij} = (A_2)_{ij} = \cdots = (A_p)_{ij} = 0 \end{aligned}$$

ARX model with zero constraints is assumed to be Granger causality which mean our data contains the zero pattern of A_k. Hence, we will reduce the number of data using vectorization. Since, we have 20,000 voxels [8] and each voxel containts a lot of data for reducing the cost computation this condition should be considered. For estimating, we can split the (11) for finding the LS closed-form solution of ARX model with zero constraints. The details of calculating will be shown in Appendix.

$$\|Y - AH - BK\|_F^2 = \sum_{s,t} (Y_{st} - (AH)_{st} - (BK)_{st})_2^2$$

$$\|Y - AH - BK\|_F^2 = \sum_{s,t} (Y_{st} - \sum_k (A)_{sk}(H)_{kt} - \sum_k (B)_{sk}(K)_{kt})_2^2 \tag{11}$$

Finally, we can provide the closed-form solution of minimize $\|Y - AH - BK\|\|_F^2$ as shown below

$$\|Y - AH - BK\|_F^2 = \|y - Gx - Fz\|_2^2 = \|y - \begin{bmatrix} G & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \|_2^2 \tag{12}$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \mathcal{G}^T\mathcal{G} & \mathcal{G}^T\mathcal{F} \\ \mathcal{F}^T\mathcal{G} & \mathcal{F}^T\mathcal{F} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{G}^T \\ \mathcal{F}^T \end{bmatrix} y \tag{13}$$

$\mathcal{G}$ and $\mathcal{F}$ is the matrix that are reduced form A and B respectively and x and z are the new coefficients which will be estimated for ARX model with zero constraints. The solution will exist if and only if matrix $\begin{bmatrix} \mathcal{G}^T\mathcal{G} & \mathcal{G}^T\mathcal{F} \\ \mathcal{F}^T\mathcal{G} & \mathcal{F}^T\mathcal{F} \end{bmatrix}$ is invertible and it has full rank. Matrix $\mathcal{G}$ and $\mathcal{F}$ should be full rank and they can't the same matrix.

3.2 The description of the data neuroimage (2013)

The data neuroimage (2013), we obtain from [7]. All participants were right-handed healthy adults and they are 22-30 years old. The experiment for recording the data divides in two parts which is shown in Figure 8 and runs during 240 s with repetition time (TR) 100 ms. Then, we got the 2400 columns which is the data at each time point.

The first is called Visual → Motor (VM) which is the main experiment. The VM experiment asks subjects to push a button when the subjects see the picture as quickly as possible. The picture will be random as uniform distribution and varying 4-16 s which the average is 10 s. If the picture shows on right hemifield, the subjects will push the button by right hand and if the picture shows on left hemifield, the subjects will push the button by left hand.

The second is called Motor → Visual (MV) which is the control experiment. The MV experiment asks subjects to push a button after hearing a 200 ms tone pip at the differential frequency as 1 kHz for left hand and 4 kHz for right hand. The tone pip is defined to be uniform distribution and varying 4-16 s which the average is 10 s like VM experiment. After the subjects push the button, the picture will display the same side as the subjects push the button by hand side. The latency between the button press and the onset of visual stimulation was 158 ms for the right and 220 ms for the left hand. This experiment is designed to check VM experiment in term of the opposite way. In the VM experiment, visual cortex works before motor cortex but motor cortex works before visual cortex for the MV experiment.

An exogenous input is included only for improving the prediction and control of some important parameter if and only if the exogenous input is known and will be useful for the prediction [9]. Then, ARX models are more suitable for fitting the data than AR models. This innovation is also helpful for characterization of the time series which is a very high dimension such as fMRI data whose obtains from [7]. So, ARX models are applied to estimate the effective connectivity of human brain in our experiments.

In data set of VM experiment in neuroimage (2013), exogenous inputs come from condition of visual stimuli. The value equals to "1" if the input stimuli shows left visual hemifield. The value equals to 2' if the input stimuli shows right visual hemifield. Otherwise, the value is "0" when there is no any stimulus. We will decompose exogenous input into two dimension for left and right hemifield ($u \in \mathbf{R}^2$). Since timing of each visual stimulus was randomized with a uniform distribution of interstimulus intervals (ISIs) varying from 4 to 16s (mean 10s) [7], it seems that inputs are pseudo random sequences in estimation. For left hemifield input, it is random sequences of unit interval [0,1]. The intervals are zero except intervals that show left visual stimuli equals to "1". For right hemifield input, it is random sequences of interval [0,2]. Other intervals are zero except intervals that show right visual stimuli equal to "2". Similar to VM experiment, exogenous inputs of MV experiment are two dimensions on condition of left or right hand push the button. They were presented randomly too. Therefore, we will use these exogenous input for our ARX model.

4 Experiments

4.1 Preprocessing on fMRI data

In this experiment, we try to remove the noise of BOLD signals. The BOLD signal in fMRI data reflects both neuronal activations and global physiological fluctuations. The main physiological processes seen in BOLD data are physiological low frequency oscillations, respiration, and cardiac pulsation. These different physiological processes confound the BOLD signals in different ways. The high frequency physiological signals (*i.e.*, from respiration and cardiac pulsation) are aliased into the low frequency band, making it hard to study the individual effect of these physiological processes on BOLD signal. Thus, we use the spectral method to find the reference signals.

Spectral method try to remove different physiological signals from BOLD signal, the power spectrum of BOLD signal at each voxel was calculated for each participant. Without external reference signal, its difficult to accurately estimate cardiac and respiratory cycles. However, we consider to search all time series for the one with the highest percentage of power spectra density between 0.8 Hz and 1.5 Hz as a "pseudo cardiac signal and the one with the highest percentage power between 0.16 Hz and 0.3 Hz as a pseudo respiratory signal. We consider this time series is pseudo cardiac or respiratory signal which influenced by BOLD signal. This experiment we use `DRIFTER` matlab toolbox which downloads from `http://becs.aalto.fi/en/research/bayes/drifter/`.

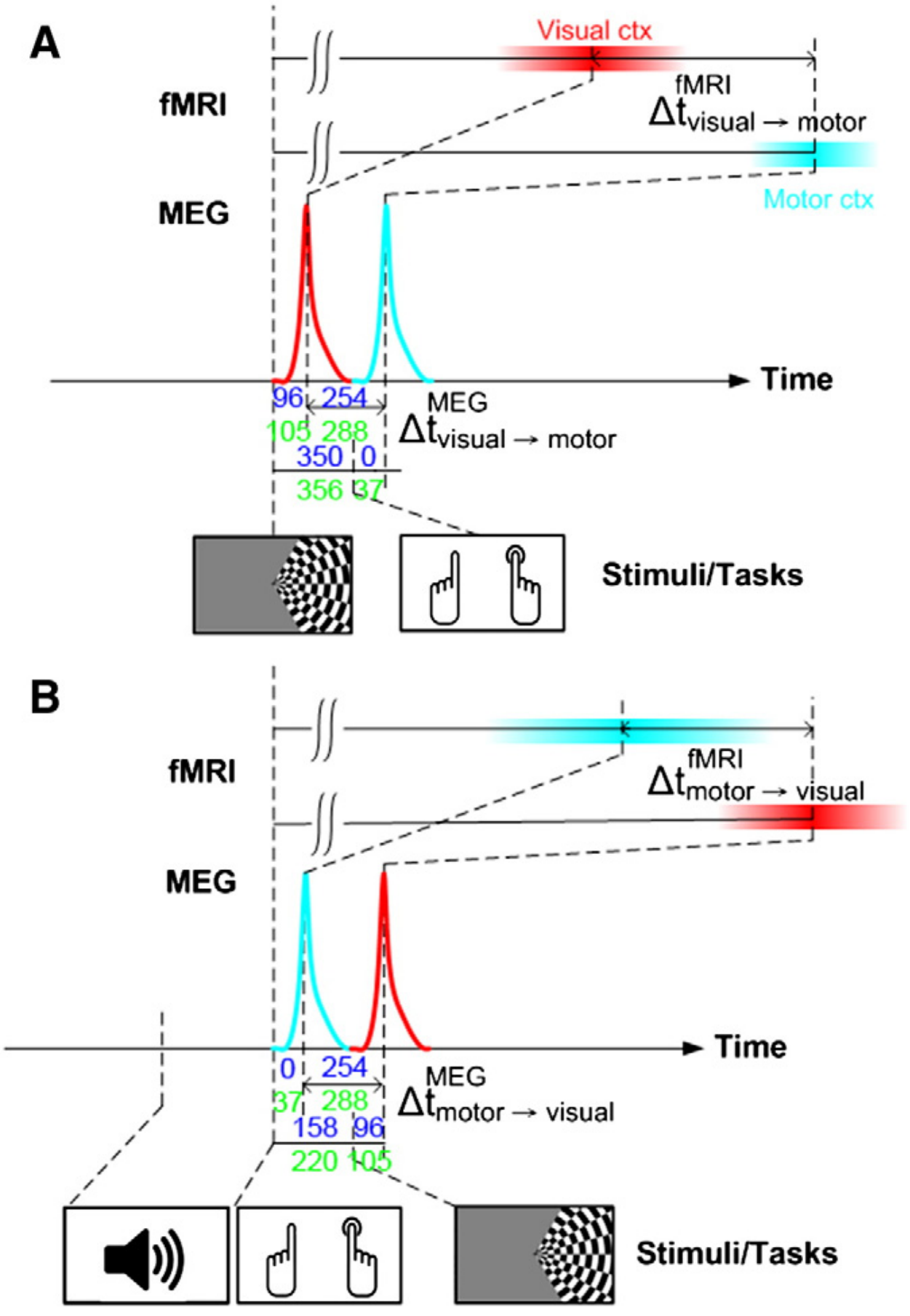


Figure 8: Experimental design of fMRI in NeuroImage2013 [7]

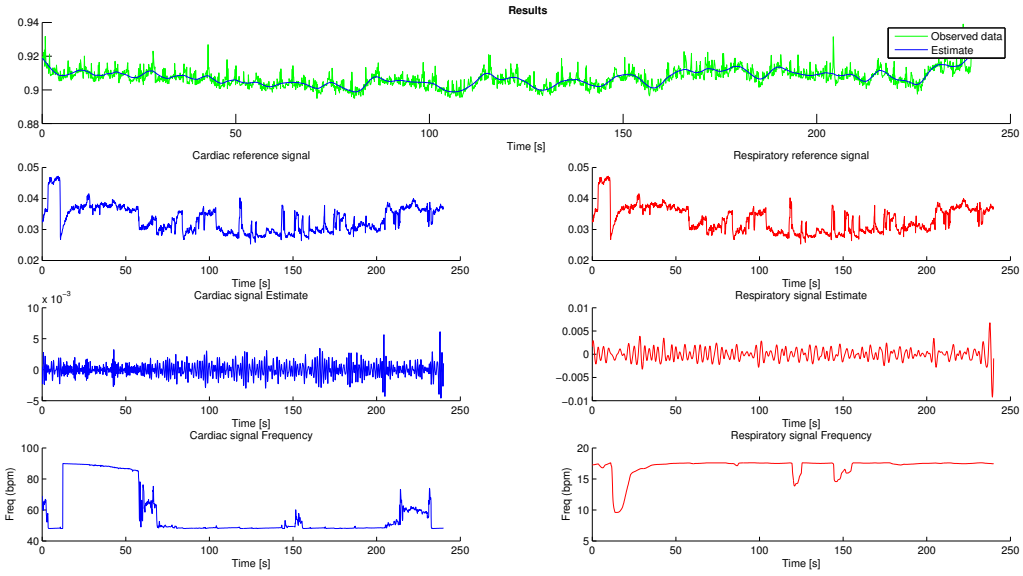
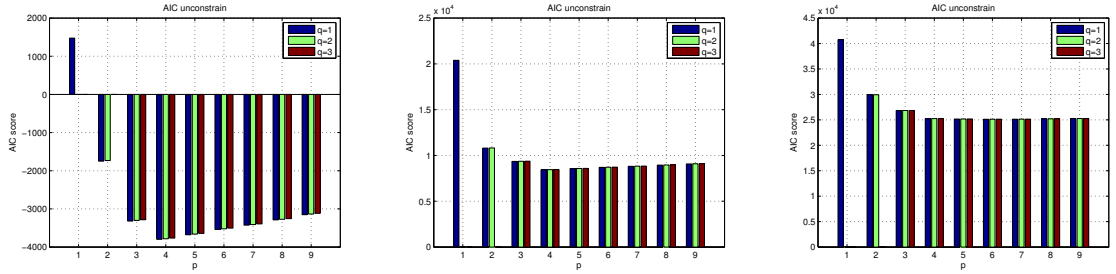


Figure 9: The example data of observed data and estimate data after remove cardiac and respiratory signals



AIC with noise variance 0.5 AIC with noise variance 0.7 AIC with noise variance 1.0

Figure 10: Bar graphs present AIC of ARX unconstraint with dense A.

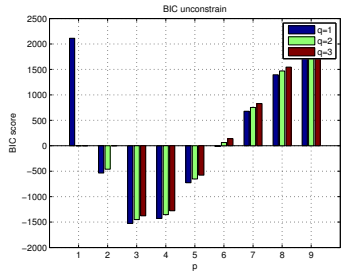
The SPM toolbox is an implement of the DRIFTER algorithm, which is a Bayesian method for physiological noise modeling and removal allowing accurate dynamical tracking of the variations in the cardiac and respiratory frequencies by using Interacting Multiple Models (IMM), Kalman Filer (KF) and Rauch-Tung-Striebel (RTS) smoother algorithms. The result from this trial will be shown in Figure 9 which the example of fMRI data run into SPM toolbox for finding the cardiac and respiratory signals. After finished this process, we will get the clean BOLD signals from fMRI data sets and will be used to do the next experiment.

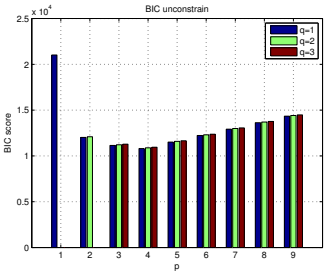
4.2 Least-square estimation of ARX models

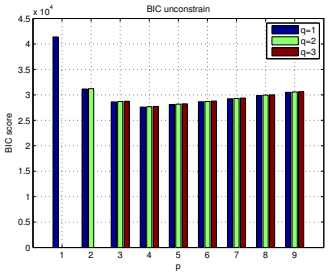
Experiment of model selection by AIC and BIC

The experiment started from generating data with order $p = 4$ and $q = 2$ and finding AIC and BIC values of all possible pairs of p, q for indicating which orders have a good fitting on ARX model. This experiment aims to examine whether AIC and BIC were able to choose the correct casual orders, $p = 4$ and $q = 2$ while varying noise variance and density of matrix A.

From the above results, we saw that each q are scanty affected in AIC and BIC value because the large number of measured data y are dominant over the small number of input u. For each q, the AIC and BIC values significantly varied when $p = 1$ to 4. For p more than 4, they slightly

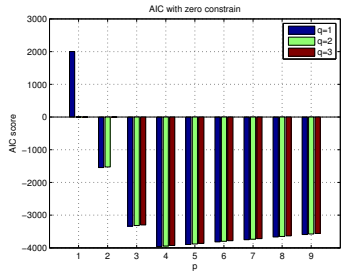


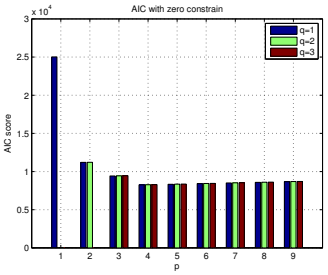


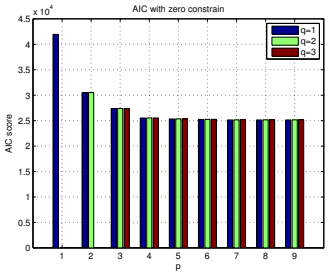


BIC with noise variance 0.5 BIC with noise variance 0.7 BIC with noise variance 1.0

Figure 11: Bar graphs present BIC of ARX unconstraint with dense A.

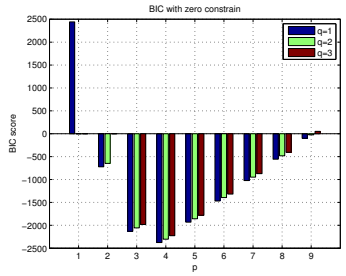


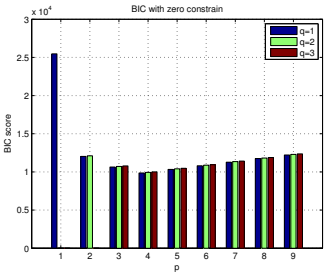


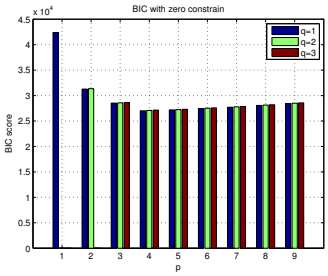


AIC with noise variance 0.5 AIC with noise variance 0.7 AIC with noise variance 1.0

Figure 12: Bar graph presents AIC of ARX zero constraint with dense A.

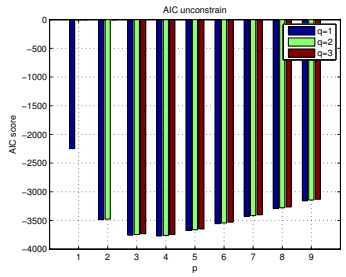


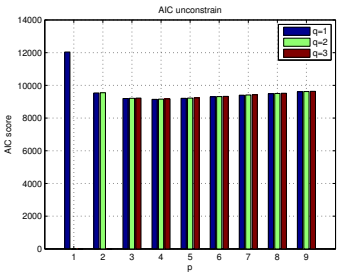


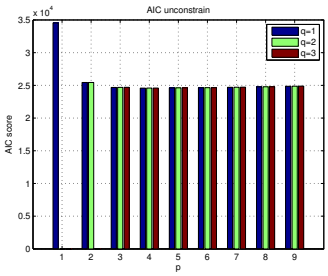


BIC with noise variance 0.5 BIC with noise variance 0.7 BIC with noise variance 1.0

Figure 13: Bar graphs present BIC of ARX zero constraint with dense A.

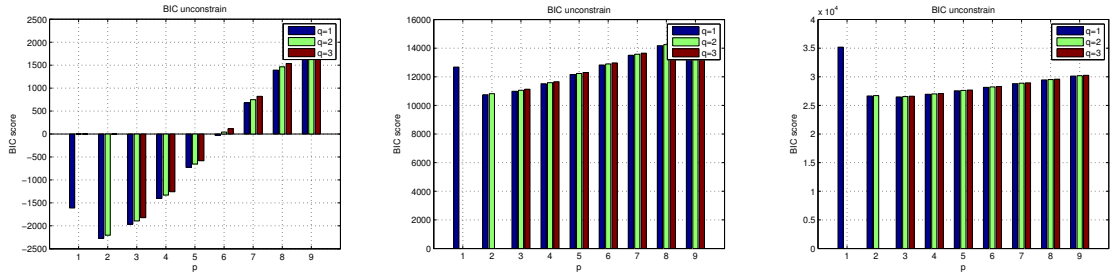






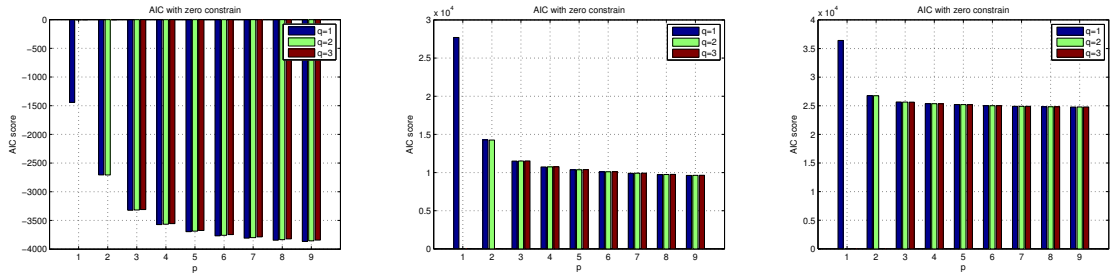
AIC with noise variance 0.5 AIC with noise variance 0.7 AIC with noise variance 1.0

Figure 14: Bar graphs present AIC of ARX unconstraint with sparse A.



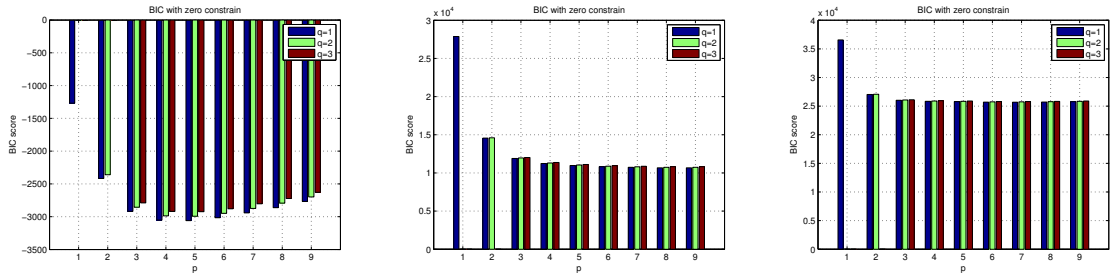
BIC with noise variance 0.5 BIC with noise variance 0.7 BIC with noise variance 1.0

Figure 15: Bar graphs present BIC of ARX unconstraint with sparse A.



AIC with noise variance 0.5 AIC with noise variance 0.7 AIC with noise variance 1.0

Figure 16: Bar graph presents AIC of ARX zero constraint with sparse A.



BIC with noise variance 0.5 BIC with noise variance 0.7 BIC with noise variance 1.0

Figure 17: Bar graphs present BIC of ARX zero constraint with sparse A.

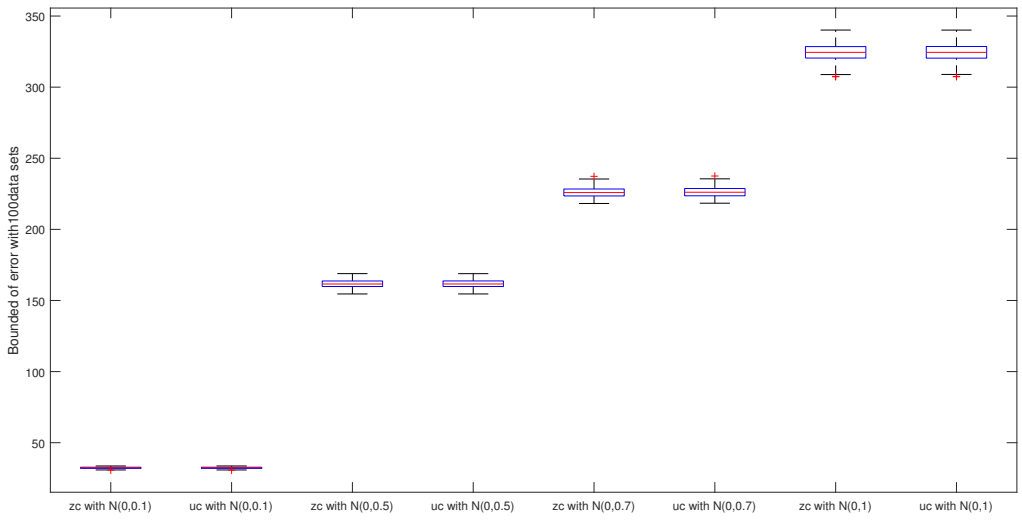


Figure 18: The norm error of 100 data sets which is generated with noise variance equals to $\mathcal{N}(0, 0.1)$, $\mathcal{N}(0, 0.5)$, $\mathcal{N}(0, 0.7)$, and $\mathcal{N}(0, 1)$ at $p = 4$, $q = 2$ and $N = 2400$ where the red line displays the median values of data sets, horizontal black line displays the bounds of data sets, blue box contains 50% of middle data sets, below vertical dash line displays 25% of the first data respect to the quartiles of a ranked set, above vertical dash line displays 25% of the last data and red cross displays outlier point.

varied. The casual order $p = 4$ and q =1 are chosen as the mode of order giving the least AIC and BIC value for each table.

Experiment of random data for testing the estimation with different noise variance

For the section of experiment, we generate the random data for testing the estimation of our ARX models. With writing our code of ARX model, we produce the experiment on our model by estimation of data which is generated in MATLAB via random data. The random data is defined to be normal distribution with zero mean and variance is varied to be 0.1, 0.5, 0.7 and 1. The random data that we generate is $y \in \mathbf{R}^{10}$ and $u \in \mathbf{R}^2$ and is defined to be stationary by given the condition. The random data sets is generated to be 100 and 200 data sets. The results of this experiments descibe the performance of estimation by our models. The details of MATLAB code that we use to find the results will be shown in Appendix.

To show the performance between models, the 100 and 200 data sets of y are generated with noise of variance which has been varied distribution as $\mathcal{N}(0, 0.1)$, $\mathcal{N}(0, 0.5)$, $\mathcal{N}(0, 0.7)$ and $\mathcal{N}(0, 1)$. With our assumption, if y is generated with high noise variance, the error between y and y_{est} will be much more than y with small noise variance. To verify the assumption, the comparison of norm error is produced with difference noise variance. The data sets contain 2400 samplings (N = 2400). As shown in 18 and 19 are the comparison of norm error which is presented by box plot.

The results of our experiments can conclude that the noise variance has affected to estimation of both techniques. As shown in the Figure 18 and 19 can conclude that the noise variance equals to 1 has higher values of bounded error than others. Likewise, y data sets with noise variance 0.7, 0.5 and 0.1 have the values of bounded error respective decrease.

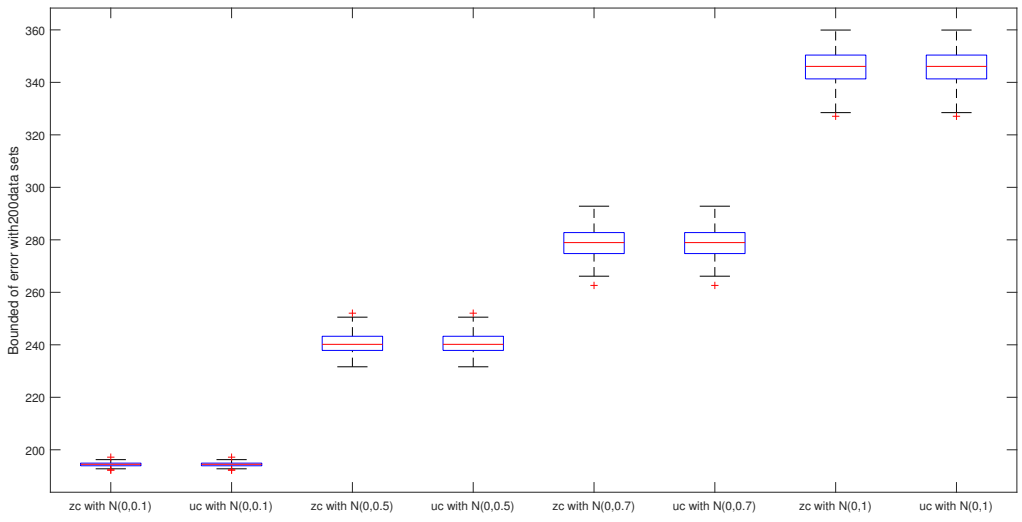


Figure 19: The norm error of 200 data sets which is generated with noise variance equals to $\mathcal{N}(0, 0.1)$, $\mathcal{N}(0, 0.5)$, $\mathcal{N}(0, 0.7)$, and $\mathcal{N}(0, 1)$ at $p = 4$, $q = 2$ and $N = 2400$ where the red line displays the median values of data sets, horizontal black line displays the bounds of data sets, blue box contains 50% of middle data sets, below vertical dash line displays 25% of the first data respect to the quartiles of a ranked set, above vertical dash line displays 25% of the last data and red cross displays outlier point.

Experiment of ARX without constraint using the real data

We apply ARX model without constrain to the data of neuroimage (2013). We observe the estimate data is closely to the observed data at the beginning time but the estimate data is unstable for all data sets as show in Figure 20. The norm of error also increasing too as shown in Figure 21. We assume this problem come from the coefficient in $\hat{A}$ is unstable, which has poles of transfer function are outside of the unit circle.

5 Conclusions

This term project focuses on the ARX model estimation of BOLD signals of each region in brain. First, we generated initial y for p terms, initial u for N terms and also generated a chosen-casual-order model, matrix A and B, under zero-constrained condition, as shown in section 3.1, from the Granger casuality concept. Then, we compute the further y by (2) until we have N sampling of y. Using all known y and u, we estimated A and B under both unconstrained and zero-constrained conditions under assumption that zero's positions in matrix A and B are unknown.

Experiment variance of noise - variance of noise affected on our model estimation because the estimation did not use any information from noise to be improved - As the result of no negative feedback system for gaining infomation from noise, more variance of noise in output causes more error of estimation

In casual order estimation section, the mode of casual order of AR and X, or (p, q), estimated under each constrained condition by each informations criterion are (4,1) whose AIC and BIC are close to the actual casual order (4,2). The result $q = 1$ may imply that there is some a casual-order-(4,1) model which is equivalent to the casual-order-(4,2) model or the second order of input is possibly not nessessary.

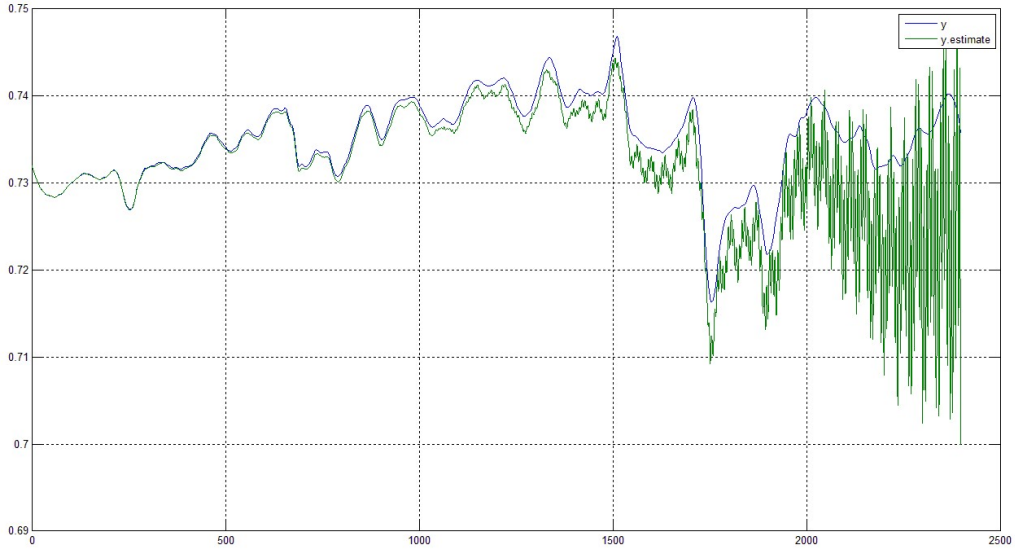


Figure 20: This figure shows the comparison between observed data (the blue line) and estimate data (the green line) of 1 voxel.

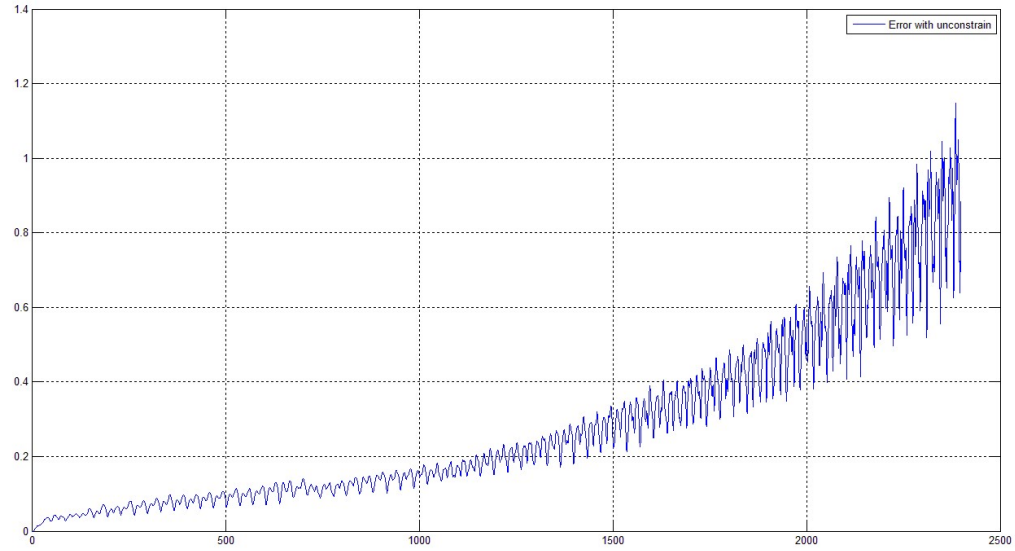


Figure 21: Y norm of error is periodic and increasing by time.

Lastly, we prepared the real fMRI data for estimation by removing respiratory and cardiac noise in section 4.1 and illustrated of using real data for estimation in section 4.2.

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Appendices

A ARX model

In this section, we want to extend why we get $\|Y - \theta H\|_F^2$ as (3).

$$\begin{bmatrix} y(p+1) \cdots y(N) \end{bmatrix} = \begin{bmatrix} A_1 & \cdots & A_p & B_1 & \cdots & B_P \end{bmatrix} \begin{bmatrix} y(p) & \cdots & y(N-1) \\ \vdots & \ddots & \vdots \\ y(1) & \cdots & y(N-p) \\ u(p) & \cdots & u(N-1) \\ \vdots & \ddots & \vdots \\ u(p-q) & \cdots & u(N-q) \end{bmatrix} + \begin{bmatrix} e(p+1) \cdots e(N) \end{bmatrix}$$

$$Y = \theta H + e \tag{14}$$

B ARX with zero constraints

From (11), we want to minimize $\|Y - AH - BK\|_F^2$. Then, vectorization is used to reduce the number of data and form a new matrix for replacing the summation term as shown below

$$\begin{aligned}\|Y - AH - BK\|_F^2 &= \sum_{s,t} (Y_{st} - (AH)_{st} - (BK)_{st})_2^2 \\ &= \sum_{s,t} (Y_{st} - \sum_{k=1}^{np} (A)_{sk}(H)_{kt} - \sum_{k=1}^{mq} (B)_{sk}(K)_{kt})_2^2\end{aligned}$$

$$\begin{aligned}A &= \begin{bmatrix} A_1 & A_2 & \cdots & A_P \end{bmatrix} \in \mathbb{R}^{n \times np} \quad where \quad A_i \in \mathbb{R}^{n \times n}, (i = 1, 2, \cdots, p) \\ B &= \begin{bmatrix} B_1 & B_2 & \cdots & B_q \end{bmatrix} \in \mathbb{R}^{n \times mq} \quad where \quad B_i \in \mathbb{R}^{n \times m}, (i = 1, 2, \cdots, q) \\ H &= \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_p \end{bmatrix} \in \mathbb{R}^{np \times N} \quad where \quad H_i \in \mathbb{R}^{n \times N}, (i = 1, 2, \cdots, p) \\ K &= \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_q \end{bmatrix} \in \mathbb{R}^{mq \times N} \quad where \quad K_i \in \mathbb{R}^{m \times N}, (i = 1, 2, \cdots, q)\end{aligned}$$

$$\sum_{k=1}^{np} (A)_{sk}(H)_{kt} = \sum_{k=1}^{n} \begin{bmatrix} (H_1)_{kt} & (H_2)_{kt} & \cdots & (H_p)_{kt} \end{bmatrix} \begin{bmatrix} (A_1)_{sk} \\ (A_2)_{sk} \\ \vdots \\ (A_p)_{sk} \end{bmatrix} = \sum_{k=1}^{n} \mathcal{H}_{k,t} X_{sk}$$

$$\sum_{k=1}^{mq} (B)_{sk}(K)_{kt} = \sum_{k=1}^{m} \begin{bmatrix} (K_1)_{kt} & (K_2)_{kt} & \cdots & (K_q)_{kt} \end{bmatrix} \begin{bmatrix} (B_1)_{sk} \\ (B_2)_{sk} \\ \vdots \\ (B_q)_{sk} \end{bmatrix} = \sum_{k=1}^{m} \mathcal{K}_{k,t} Z_{sk}$$

$\Rightarrow$

$$\begin{bmatrix} Y_{S,1} \\ Y_{S,2} \\ \vdots \\ Y_{s,N} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} & \cdots & \mathcal{H}_{1,n} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} & \cdots & \mathcal{H}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{N,1} & \mathcal{H}_{N,2} & \cdots & \mathcal{H}_{N,n} \end{bmatrix} \begin{bmatrix} X_{s,1} \\ X_{s,2} \\ \vdots \\ X_{s,n} \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{1,1} & \mathcal{K}_{1,2} & \cdots & \mathcal{K}_{1,m} \\ \mathcal{K}_{2,1} & \mathcal{K}_{2,2} & \cdots & \mathcal{K}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{N,1} & \mathcal{K}_{N,2} & \cdots & \mathcal{K}_{N,m} \end{bmatrix} \begin{bmatrix} Z_{s,1} \\ Z_{s,2} \\ \vdots \\ Z_{s,m} \end{bmatrix}$$

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_{1,1} & \mathcal{H}_{1,2} & \cdots & \mathcal{H}_{1,n} \\ \mathcal{H}_{2,1} & \mathcal{H}_{2,2} & \cdots & \mathcal{H}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{N,1} & \mathcal{H}_{N,2} & \cdots & \mathcal{H}_{N,n} \end{bmatrix}, \mathcal{K} = \begin{bmatrix} \mathcal{K}_{1,1} & \mathcal{K}_{1,2} & \cdots & \mathcal{K}_{1,m} \\ \mathcal{K}_{2,1} & \mathcal{K}_{2,2} & \cdots & \mathcal{K}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{N,1} & \mathcal{K}_{N,2} & \cdots & \mathcal{K}_{N,m} \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} Y_{1,2} \\ \vdots \\ Y_{1,N} \\ Y_{2,1} \\ \vdots \\ Y_{n,N} \end{bmatrix} = (I_n \otimes \mathcal{H}) \begin{bmatrix} X_{1,1} \\ \vdots \\ X_{1,n} \\ X_{2,1} \\ \vdots \\ X_{n,n} \end{bmatrix} + (I_n \otimes \mathcal{K}) \begin{bmatrix} Z_{1,1} \\ \vdots \\ Z_{1,n} \\ Z_{2,1} \\ \vdots \\ Z_{n,n} \end{bmatrix}$$

$$G = I_n \otimes \mathcal{H}, F = I_n \otimes \mathcal{K}$$

$\Rightarrow$ We got,

$$\|Y - AH - BK\|_F^2 = \|y - Gx - Fz\|_2^2 \tag{15}$$

$$= \|y - \begin{bmatrix} G & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \|_2^2 \tag{16}$$

$$\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \mathcal{G}^T\mathcal{G} & \mathcal{G}^T\mathcal{F} \\ \mathcal{F}^T\mathcal{G} & \mathcal{F}^T\mathcal{F} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{G}^T \\ \mathcal{F}^T \end{bmatrix} y \tag{17}$$

C Programming code

C.1 The matlab code for finding the cardiac and respiratory signals

This part is a matlab code for finding the cardiac and respiratory signals from fMRI data sets using DRIFTER toolbox. This is available download from http://becs.aalto.fi/en/research/bayes/drifter/.

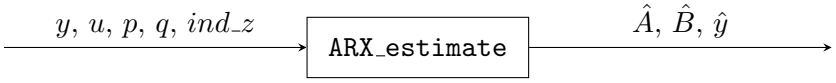
```
%Obtain the periodogram using fft. The signal is real-valued and has odd length.
%Because the signal is real-valued, you only need power estimates for the positive or negativ
%In order to conserve the total power,
%multiply all frequencies that occur in both sets the positive and negative frequencies by a
%Zero frequency (DC) and the Nyquist frequency do not occur twice.
load KuoPC_mv_02.mat;
%Load data(data contains data.voxels and data.input)
rawdata = KuoPC_mv_02.voxels;
N = 2399;                          %number of time points
T = linspace(0,240,N);
resp_max = 0;
card_max = 0;
for i = 1:length(rawdata)
    %----------------- power spetra of time series ---------------------
    x = rawdata(i,:);
    N = length(x);
    Fs = 1/(T(2)-T(1));
    xdft = fft(x);
    xdft1 = xdft(1:N/2+1);
    psdx = (1/(2*pi*N)) * abs(xdft1).^2;       %energy spectral density
    psdx(2:end-1) = 2*psdx(2:end-1);
    freq = 0:Fs/length(x):Fs/2;
    %-----------------------------------------------------------------

    %--------------------- find graph area --------------------------
    power = 0;
    resp = 0;
    card = 0;
    for j = 1:(N/2+1)
    %consider only positive freq and zero freq do not occur twice
        freq_sam = freq(j+1)-freq(j);
        power = power + (freq_sam*(psdx(j)+psdx(j+1))/2);
        %freq range from 0.16 to 0.3 consider as a respiratory signals
        if (0.16 < freq(j) && freq(j) <= 0.3)
            resp = resp + (freq_sam*(psdx(j)+psdx(j+1))/2);
        %freq range from 0.8 to 1.5 consider as a cardiac signals
        elseif (0.8 < freq(j) && freq(j) <= 1.5)
            card = card + (freq_sam*(psdx(j)+psdx(j+1))/2);
        end
    end
    %-----------------------------------------------------------------

    %------------------------ find maximum --------------------------
    if(resp/power>resp_max)
        resp_max = resp/power;
        resp_signal = x;
```

```
        num_vox_resp = i;
    end
    if(card/power>card_max)
        card_max = card/power;
        card_signal = x;
        num_vox_card = i;
    end
    fprintf('%d %d %d %d %d\n',resp_max,card_max,num_vox_resp,num_vox_card,i)
    %-----------------------------------------------------------------
end
%then we apply the DRIFTER for remove respiratory and cardiac signals from our data
```

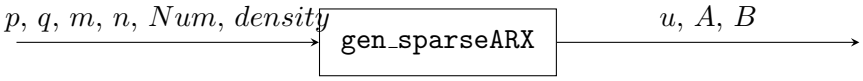
C.2 Matlab code of ARX models

This part is matlab code which is used to find matrix coefficients as (3) for ARX without constraint and (13) for ARX with zero constraint. We apply these code for finding the results in experiment parts. Our codes consist of:

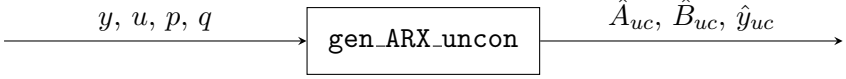
- `ARX_estimate` function is generated to estimate both ARX model without constraint or with zero constraint. This function presented $\hat{y}$, $\hat{A}$ and $\hat{B}$.



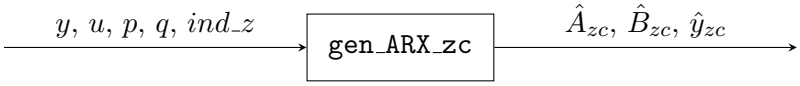
- `gen_sparseARX` function is used to find A, B, ind_z, ind_nz where ind_z is the zero and ind_nz is the non-zero elements of A. When we put n, m, p, q and $density$ of A.



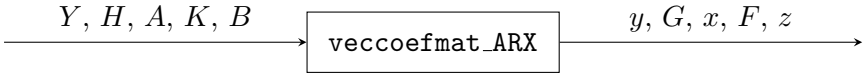
- `est_ARX_uncon` function is used to find $\hat{A}$ and $\hat{B}$ for ARX with unconstraint when we put y, u, p, q.



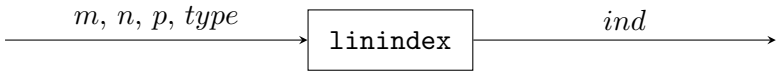
- `est_ARX_zc` function is used to find $\hat{A}$ and $\hat{B}$ for ARX with zero constraint when we put y, u, p, q and ind_z. `est_ARX_zc` consist of `veccoefmat_ARX`, `eliminate`, `include` and `form_A` function.



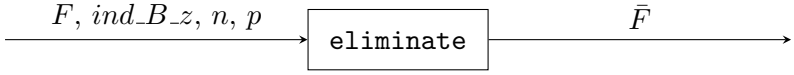
  ▶ `veccoefmat_ARX` function is used to form a new matrix fof vectorization technique as $\|Y - AH - BK\|_F^2 = \|y - Gx - Fz\|_2^2$.



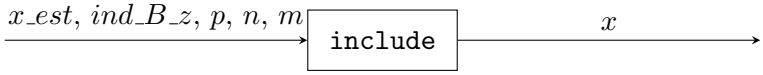
  ◦ `linindex` is a sub-function of `veccoefmat_ARX` that gives the linear indices of the vectorized 3D array.



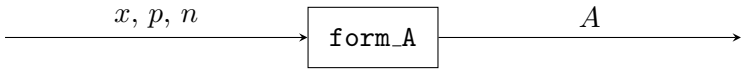
▶ `eliminate` function is used to change F to $\bar{F}$ where F is matrix output of vectorizing H.



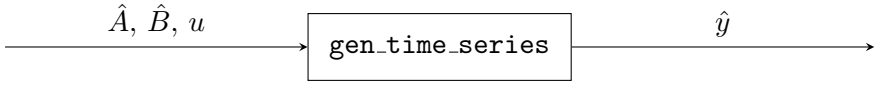
▶ `include` function is used to change x_{est} to x where x_{est} is answer of least square method.



▶ `form_A`



• `gen_time_series` function is used to form matrix $\hat{A}$ and $\hat{B}$ as ARX model
$y(t) = A_1 y(t-1) + \cdots + A_p y(t-p) + B_1 u(t-1) + B_q u(t-q) + e(t)$
where $\hat{A} = [A_1 \ A_2 \ldots A_p]$ and $\hat{B} = [B_1 \ B_2 \ldots B_q]$.



`ARX_estimate` function

```
function [ A_est, B_est, y_est ] = ARX_estimate( y, u, p, q,ind_z )
%function[ A_est, B_est, y_est ] = ARX_estimate( y, u, p, q, ind_z )
%ARX_estimate is to estimate ARX with unconstraint and zero constraint
%It is estimation ARX without constraint if there are 4 inputs
if nargin == 4
[ A_est, B_est ] = est_arx_uncon(y, u, p, q);
y1 = y(:,1);
[y_est]=gen_time_serries(A_est,B_est,u,0,y1);
end
%-------------------------------------------------------------------------
%It is estimation ARX with zero constraint if there are 5 inputs
if nargin == 5
[ A_est, B_est ] = est_arx_zc(y, u, p, q,ind_z);
y1 = y(:,1);
[y_est]=gen_time_serries(A_est,B_est,u,0,y1);
end
end
```

gen_sparseARX **function**

```
function [u,A,B] = gen_sparseARX(n,p,m,q,Num,density)
% gen_sparseARX generates a sparse vector autoregressive model with exogenous inputs
% [ind_zz,ind_nz,A,B] = gen_sparseARX(n,p,m,q,noise_var,density,Num)
% This code is generate only p in ARX Model
% ARX
% y(t) = A1*y(t-1) + A2*y(t-2) + ... + Ap*y(t-p)+ B1*u(t-1) + B2*u(t-2) + ... + Bq*u(t-q) + e
%
% 'A' represents AR coefficients A1,A2,...,Ap and is stored as a p-dimensional array
% 'B' represents X coefficients B1,B2,...,Bq and is stored as a q-dimensional array
% The input arguments are
% 'n': dimension of output
% 'p': order of 'AR' in ARX model
% 'm': dimension of input
% 'q': order of 'X' in ARX model
% 'noise_var': variance of u(t) (noise)
% 'density': the fraction of nonzero entries in AR coefficients
% 'Num': number of data points in time series
%
% The AR coefficients are sparse with a common sparsity pattern. The
% indices of nonzero entries are saved in 'ind_nz'.
%
% 'y' is a time series generated from the modAel and has size n x Num
%  y = [y(1) y(2) ... y(Num)]
%
% if p = 0, 'y' is simply a random variable. In this case, A is the
% covariance matrix of u with sparse inverse.

%% Generate u
u=rand(m,Num);
for a=1:Num
    if norm(u(:,a)) <= 1
        u(:,a)=zeros(m,1);
    elseif norm(u(:,a)) <= 2
        u(:,a)=[1;zeros(m-1,1)];
    else
        u(:,a)=[1;zeros(m-1,1)];
    end
end

%% Static case
if (p==0),
    S = sparse(2*eye(n)+sign(sprandsym(n,density)));
    [i,j]=find(S);
    S = S+sparse(ceil(max(0,-min(eig(S))))*eye(n));
    A = S\eye(n); % covariance matrix with sparse inverse
    R = chol(phi);
    y = R'*randn(n,Num); % y reduces to a random variable with covariance 'phi'
    return;
end
```

```
%% Randomize AR coefficients
MAX_EIG = 1;
diag_ind = find(eye(n));
k = length(diag_ind);
diag_ind3D = kron(n^2*(0:p-1)',ones(k,1))+kron(ones(p,1),diag_ind);

A = zeros(n,n,p);
S = sprand(n,n,density)+eye(n);
ii = 0;
while MAX_EIG,
    ii = ii+1;
    for k=1:p,
        A(:,:,k) = 0.1*sprandn(S);
    end
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    poles = -0.7+2*0.7*rand(n,p); % make the poles inside the unit circle
    characeq = zeros(n,p+1);
    for jj=1:n,
        characeq(jj,:) = poly(poles(jj,:)); % each row is [1 -a1 -a2 ... -ap]
    end
    aux = -characeq(:,2:end);
    A(diag_ind3D) = aux(:); % replace the diagonal entries with stable polynomial

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    AA =[];
    for k=1:p,
        AA = [AA A(:,:,k)];
    end
    AA = sparse([AA ; [eye(n*(p-1)) zeros(n*(p-1),n)]]);
    if max(abs(eigs(AA))) < 1
        MAX_EIG = 0;
    end
end
abs(eigs(AA))
ii
B=rand(n,m,q);
```

est_arx_uncon **function**

```
function [ A, B, y_est ] = est_arx_uncon(y, u, p, q)
% function [ A, B, y_est ] = est_arx_uncon(y, u, p, q)
% This function is generate
%y is output data with y=[y(1) y(2) ... y(N)]
%-----------------------------------------------------------------------%
    [n,N]=size(y);
    [row_u,col_u]=size(u);
    %check error input
%Compute Y of measured output
    Y = y(:,p+1:N);
%Compute matrix of causal sequence of output H_y
```

```
    H_y=[];
    for i= p:-1:1
        H_y=[H_y ;y(:,i:end-p-1+i)];
    end
%Compute matrix of causal sequence of input H_u
    H_u=[];
    for i= p:-1:p-q+1
        H_u=[H_u;u(:,i:end-p-1+i)];
    end
%Compute matrix H
    H = [H_y; H_u];
%Estimate coefficient C
    C = Y*pinv(H);
%Separate coefficient of input and coefficient of ouput
    A0 = C(:,1:n*p);
    B0 = C(:,n*p+1:end);
    for i=1:p
        A(:,:,i)=A0(:,1+n*(i-1):n*i);
    end
    for i=1:q
        B(:,:,i)=B0(:,1+row_u*(i-1):row_u*i);
    end
%Reshape matrix A into 2D
    row_A2D = size(A,1);
    col_A2D = size(A,2)*size(A,3);
    A2D = reshape(A,[row_A2D, col_A2D]);
%Reshape matrix B into 2D
    row_B2D = size(B,1);
    col_B2D = size(B,2)*size(B,3);
    B2D = reshape(B,[row_B2D, col_B2D]);
%estimated output
y_est = [A2D, B2D]*H;
end
```

est_arx_zc function

```
function [ A, B, y_est ] = est_arx_zc(y, u, p, q,ind_z)
% function [ A, B, y_est ] = est_arx_zc(y, u, p, q,ind_nz)
% This function estimate parameter in ARX model by using least square and
% zero constain
%
% input
% 'y' is array of input
% 'u' is array of input
% 'p' is number of lag of in estimate AR factor in ARX
% 'q' is number of lag of in estimate X factor in ARX
% 'ind_z' are index of A is zero in AR factor
% output
% 'A' are matrices of AR factor in ARX model
% 'B' are matrices of X factor in ARX model
% 'y_est' are estimated output from zero constraint
%%
```

```
[row_y,col_y]=size(y);
[row_u,col_u]=size(u);
%% gen Y H K
Y=y(:,p+1:end);

    H=[];
    for i= p:-1:1
        H=[H ;y(:,i:end-p+i-1)];
    end
    K=[];
    for i= p:-1:p-q+1
        K=[K;u(:,i:end-p+i-1)];
    end
A=rand(row_y,row_y*p);
B=rand(row_y,row_u*q);
%% vectorization and solve least square
[y_vec,G,x,F,z] = veccoefmat_ARX(Y,H,A,K,B,row_u);
[row_z,col_z]=size(z);
[G_bar]=eliminate(G,ind_z,row_y,p);
xz_ls=[G_bar F]\y_vec;
[row_xz,col_xz]=size(xz_ls);
x_ls=xz_ls(1:row_xz-row_z,:);
z_ls=xz_ls(row_xz-row_z+1:end,:);
x=include(x_ls,ind_z,p,row_y,row_y);
%% create A,B form x,z and check A,B%%%%%%%%%%%%%%%%%%%%
A=form_A(x,p,row_y);
B=form_A(z_ls,q,row_y);
%% create estmated y%%%%%%%%%%%%%%%%%%%%
%Reshape matrix A into 2D
    row_A2D = size(A,1);
    col_A2D = size(A,2)*size(A,3);
    A2D = reshape(A,[row_A2D, col_A2D]);
%Reshape matrix B into 2D
    row_B2D = size(B,1);
    col_B2D = size(B,2)*size(B,3);
    B2D = reshape(B,[row_B2D, col_B2D]);
%estimated output
y_est = [A2D, B2D]*[H; K];
end
```

include **function**

```
function[x]=include(x_est,ind_B_z,p,n,m)
% this code is change x_est to x
% x_est is answer of least square method
% x is x_est that is included zeros elements
% x in R^nmp
% ind_B_nz is index of B that B(ind_B_nz) is not zero
% n is number of length of output y
% q is number of lag in matrix
    x=[];
    [I,J]=ind2sub([n m],ind_B_z);
```

```
    [row_I,col_I]=size(I);
    %% source I,J depend on J %%%%%%%%%%%%%%%%%%%%%%%%%
    for i=1:row_I
        for j=1:i-1
            fn_i=m*(I(i)-1)+J(i);
            fn_j=m*(I(j)-1)+J(j);
            if fn_i < fn_j
                a=I(i);
                b=J(i);
                for k = i-1:-1:j
                    I(k+1)=I(k);
                    J(k+1)=J(k);
                end
                I(j)=a;
                J(j)=b;
            end
        end
    end
    %% create x
    k=1;
    l=1;
    for i=1:n*m
       z_element=m*(I(k)-1)+J(k);
        if i~=z_element
            x=[x; x_est(p*(l-1)+1:p*l,:)];
            l=l+1;
        else
            x=[x;zeros(p,1)];
            k=k+1;
            if k > row_I
                x=[x; x_est(p*(l-1)+1:end,:)];
                break
            end
        end
    end
end
```

elimainate **function**

```
function[F_bar]=eliminate(F,ind_B_z,n,q)
% this code is change F to F_bar
% F is matrix output of vectorizing H
% F is R^nN*n*m*p
% F_bar is F that is elimated column relate x_i = 0
% ind_B_nz is index of B that B(ind_B_nz) is not zero
% n is number of length of output y
% q is number of lag in matrix

    [row_F,col_F]=size(F);
    m=floor(col_F/(n*q));
    [I,J]=ind2sub([n m],ind_B_z);
    [row_I,col_I]=size(I);
    %% source I,J depend on J %%%%%%%%%%%%%%%%%%%%%%%%%
```

```
    for i=1:row_I
        for j=1:i-1
            fn_i=m*(I(i)-1)+J(i);
            fn_j=m*(I(j)-1)+J(j);
            if fn_i < fn_j
                a=I(i);
                b=J(i);
                for k = i-1:-1:j
                    I(k+1)=I(k);
                    J(k+1)=J(k);
                end
                I(j)=a;
                J(j)=b;
            end
        end
    end
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %% create F_bar %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    F_bar=[];
    k=1;
    for i=1:n*m
        z_element=m*(I(k)-1)+J(k);
        if i~=z_element
            F_bar=[F_bar F(:,q*(i-1)+1:q*i)];
        else
            k=k+1;
            if k > row_I
                F_bar=[F_bar F(:,q*(i)+1:end)];
                break
            end
        end
    end
end
```

`form_A` **function**

```
function[A]=form_A(x,p,n)
    [row_x,col_x]=size(x);
    nm=floor(row_x/p);
    m=floor(nm/n);
    for i=1:nm
        lin_x(:,i)=x((i-1)*p+1:i*p,:);
    end
    for i=1:p
        AA(:,i)=lin_x(i,:)';
    end
    AAA=reshape(AA,m,n,p);
    for i=1:p
        A(:,:,i)=AAA(:,:,i)';
    end
end
```

gen_time_series function

```
function[y_time]=gen_time_series(A,B,u,noise_var)
% gen_time_series generates y in term of time series
%[y_time]=gen_time_series(A,B,u,noise_var)
% This code is generate only p in ARX Model
% ARX
% y(t) = A1*y(t-1) + A2*y(t-2) + ... + Ap*y(t-p)+ B1*u(t-1) + B2*u(t-2) + ... + Bq*u(t-q) + e
% 'A' represents A_est of AR coefficients A1,A2,...,Ap and is stored as a p-dimensional array
% 'B' represents B_est of X coefficients B1,B2,...,Bq and is stored as a q-dimensional array
%
% The input arguments are
% 'A': A_est that is estimated by est_ARX_uncon or est_ARX_zc function
% 'B': B_est that is estimated by est_ARX_uncon or est_ARX_zc function
% 'u': exogenous input which is generated form main.m
% 'noise_var': variance of u(t) (noise)
%
%
[row_A,col_A,num_A]=size(A);
[row_B,col_B,num_B]=size(B);
[row_u,col_u]=size(u);
sigma = noise_var;
y1 = rand(row_A, 1);
%% check error
if row_A ~= col_A
    error('uuhiuhiuhiuh');
end
if row_A ~= row_B
    error('hoihijhojijojio');
end
if col_B ~= row_u
    error('uhoiijojoijoijo');
end
%gen time serries
y_time=[y1 zeros(row_A,col_u-1)];

    for i=2:col_u
        for j=1:num_A
            if i - j > 1
                y_time(:,i)=y_time(:,i)+A(:,:,j)*y_time(:,i-j);
            end
        end
        for j=1:num_B
            if i - j > 0
                y_time(:,i)=y_time(:,i)+B(:,:,j)*u(:,j);
            end
        end
        e = normrnd(0,sigma,[row_A 1]);
        y_time(:,i)=y_time(:,i)+ e;
    end
end
```

linindex function

```
function [ind] = linindex(m,n,p,type)
% LININDEX gives the linear indices of the vectorized 3D array
% Z = (Z1,Z2,...,Zp) where Zk is m x n
% if type == 'row' ind is the linear indices of
%     [(Z1)_{11} (Z2)_{11} ... (Zp)_{11}
%       (Z1)_{12} (Z2)_{12} ... (Zp)_{12} ...
%       (Z1)_{1n} (Z2)_{1n} ... (Zp)_{1n}
%       (Z1)_{21} (Z2)_{21} ... (Zp)_{21} ...
%       (Z1)_{2n} (Z2)_{2n} ... (Zp)_{2n} ...
%       (Z1)_{m1} (Z2)_{m1} ... (Zp)_{m1} ...
%       (Z1)_{mn} (Z2)_{mn} ... (Zp)_{mn}
%       ]
% if type == 'col' ind is the linear indices of
%     [(Z1)_{11} (Z2)_{11} ... (Zp)_{11}
%       (Z1)_{21} (Z2)_{21} ... (Zp)_{21} ...
%       (Z1)_{m1} (Z2)_{m1} ... (Zp)_{m1}
%       (Z1)_{12} (Z2)_{12} ... (Zp)_{12} ...
%       (Z1)_{m2} (Z2)_{m2} ... (Zp)_{m2} ...
%       (Z1)_{1m} (Z2)_{1m} ... (Zp)_{1m} ...
%       (Z1)_{mn} (Z2)_{mn} ... (Zp)_{mn}
%       ]

K = repmat([1:p]',m*n,1); % same as K = kron(ones(m*n,1),[1:p]');

switch type
    case 'row'
        I = kron([1:m]',ones(n*p,1));
        Jtmp = kron([1:n]',ones(p,1));
        J = repmat(Jtmp,m,1); % same as J = kron(ones(m,1),Jtmp);
        ind= sub2ind([m n p],I,J,K);
    case 'col'
        Itmp = kron([1:m]',ones(p,1));
        I = repmat(Itmp,n,1); % same as I = kron(ones(n,1),Itmp);
        J = kron([1:n]',ones(m*p,1));
        ind = sub2ind([m n p],I,J,K);
end
```