2102531 System Identification

An EEG subspace identification by using subspace method
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Abstract

Subspace identification is a tool which used for estimate state sequence and system matrices of model. The method is very useful because with only prior knowledge (y), we can estimate all unknown state space variables and parameters. The objectives of this study are to estimate the model from EEG time series data by using subspace method and to examine the estimated model with Granger causality (GC) test defined on state space equation. To achieve the objectives, we compared the Granger causality (GC) test of AR model with GC test on state space model that are generated from ground truth AR model. The result showed that the structure for GC test on state space model has the same structure as the test on AR model. Also, we compared GC test on estimated state space that are estimated from time series data generated by ground truth AR model. The result showed that the structure for GC test on estimated model is not the same structure as AR model. The estimation in subspace method is one of the cause for this error.

1 Background

1.1 EEG Model

Located at the brain and spine, Central Nervous System (CNS) is the place where neural activities occurred. This happened by potential at gap between Axon and Dendrite called Synapse by stimulated to surround environment. An electroencephalogram (EEG) is one of tools to measure brain rhythms by measuring ionic current voltage fluctuations from electrodes placed on the scalp in special position [1] that specified using international 10-20 system. Each position is labeled with a letter and a number. The letter means area that electrode lied [2]. For example, F7 means node number 7 at Frontal lobe area.
Fig.2 is the data of 100 single-channel EEG segments of 4097 samples (23.6 seconds duration) dependence on recording region and brain state.

From the raw data, EEG are spontaneously non-stationary because statistical properties of the brain processes vary over time. Also, dynamical parameters of EEG are sensitive to time scales that involved in the process to get an insight in the working of brain [4]. By using EEG to analyze human brain activity, there are many mathematical model that describe EEG model. One can be generally described by linear Autoregressive (AR) model, which expressed as [5] :

$$x(t) = \sum_{k=1}^{p} A(k)x(t-k) + u(t) \tag{1a}$$

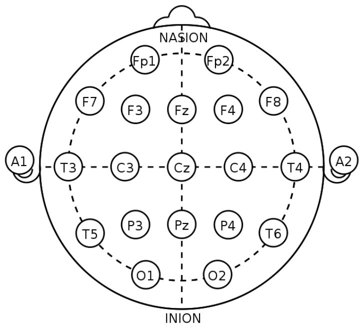


Figure 1: Electrode locations of International 10-20 system for EEG recording. The letters F,T,C,P and O stand for frontal, temporal, central, parietal, and occipital lobes, respectively.

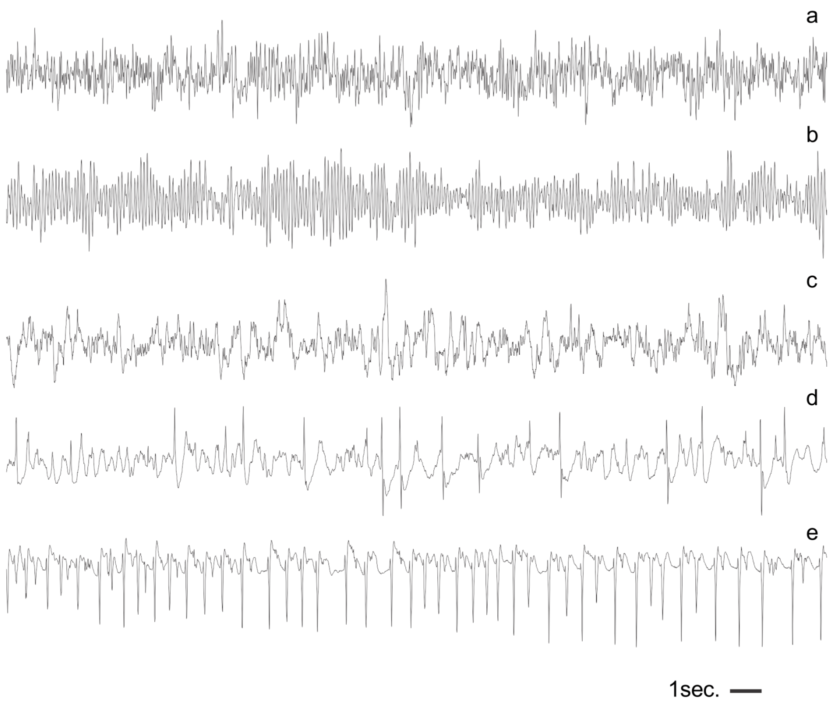


Figure 2: Raw data of EEG time series with awake state with eyes open (a) and eyes closed (b). The others were recorded during seizuring interval (c),(d) and during seizure activity (e) [3]

$$y(t) = Lx(t) + v(t) \tag{1b}$$

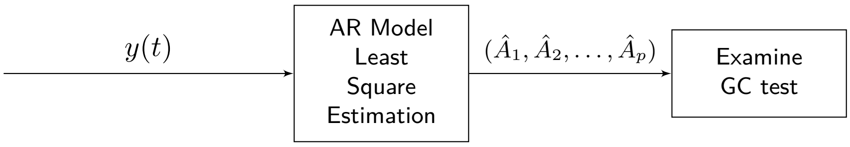
where $x \in \mathbf{R}^n$ is sample of brain source with n nodes at time t , $y \in \mathbf{R}^m$ is an EEG measurement (result show in terms of time series model from Figure 2) contains m sources at time t , $A_k \in \mathbf{R}^{n \times n}$ denotes parameters of past data , $L \in \mathbf{R}^{m \times n}$ means the lead field matrix , u and v are noise from source and noise from measurement, respectively and noise covariance matrix are given by :

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \mathbf{E}\left\{ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{\mathsf{T}} \right\} \tag{2}$$

1.2 Granger causality

Granger causality is a tool used for analyzing a brain connectivity. This tool is commonly expressed in terms of prediction error. The result of Granger causality for EEG indicates data from one part of the brain cause or does not cause to another part data. There are many approaches to examine GC Test. In this project, We focus on only two Granger causality test : GC test on AR model and GC test on state space model.

Granger causality on AR model For linear AR model, Granger causality are performed after we estimate the system matrices of AR model shown as the following diagram.



For example, $x_1(t)$, $x_2(t)$ and $x_3(t)$ in AR model have relation shown as:

$$
\begin{aligned}
x_1(t) &= \sum_{k=1}^{p} a_k x_1(t-k) + \sum_{k=1}^{p} b_k x_2(t-k) + \sum_{k=1}^{p} c_k x_3(t-k) + \varepsilon_1(t) \\
x_2(t) &= \sum_{k=1}^{p} d_k x_1(t-k) + \sum_{k=1}^{p} g_k x_2(t-k) + \sum_{k=1}^{p} h_k x_3(t-k) + \varepsilon_2(t) \\
x_3(t) &= \sum_{k=1}^{p} m_k x_1(t-k) + \sum_{k=1}^{p} n_k x_2(t-k) + \sum_{k=1}^{p} r_k x_3(t-k) + \varepsilon_3(t)
\end{aligned}
\tag{3}
$$

with covariance of noise as $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} = \mathrm{Cov}(\varepsilon)$.

Then, we assume that $x_2(t)$ is not a cause for $x_1(t)$ so the new model will be reduced and remain only $x_1(t)$ and $x_3(t)$

$$
\begin{aligned}
x_1(t) &= \sum_{k=1}^{p} a'_k x_1(t-k) + \sum_{k=1}^{p} c'_k x_3(t-k) + \varepsilon'_1(t) \\
x_3(t) &= \sum_{k=1}^{p} m'_k x_1(t-k) + \sum_{k=1}^{p} r'_k x_3(t-k) + \varepsilon'_2(t)
\end{aligned}
\tag{4}
$$

with noise covariance of reduced model $\Sigma^R = \begin{bmatrix} \Sigma^R_{11} & \Sigma^R_{13} \\ \Sigma^R_{31} & \Sigma^R_{33} \end{bmatrix}$.

After that, we examine if $x_2(t)$ has causality relation to $x_1(t)$ by determining log ratio of residual error of $x_1(t)$ for each model [6].

$$
\mathcal{F}_{x_2 \to x_1 \mid x_3} = \log \frac{\Sigma^R_{11}}{\Sigma_{11}} \tag{5}
$$

In general, $\Sigma^R_{11} > \Sigma_{11}$ because variance is minimized when data is added. From (5), if $\mathcal{F}_{x_2 \to x_1 \mid x_3} = 0$, it means $\Sigma^R_{11} = \Sigma_{11}$. Therefore, $x_2(t)$ is not cause $x_1(t)$. On the other hand, $x_2(t)$ cause to $x_1(t)$ when $\mathcal{F}_{x_2 \to x_1 \mid x_3} > 0$ because $x_1(t)$ in full model usually have more fitting than $x_1(t)$ in reduced model so that Σ^R_{11} always more than Σ_{11}. Also, the result of GC test on AR model can be derived as $(A_k)_{ij} = 0$ and examine which x_j is not a cause for x_i.

Granger causality on state space model In case of state space model Granger causality test, state space equation is

$$
z(t+1) = \mathcal{A} z(t) + w(t) \tag{6a}
$$

$$
y(t) = C z(t) + v(t) \tag{6b}
$$

In this Granger causality test, we examine if y_j is a cause for y_i by removing y_j from the model. To remove y_j from the model, we force j^{th} column of C from (6) be zero so that full model become reduced model. Then, determine residual error of both models. Finally, we determine log ratio of residual error of x_i for each model. [7] :

$$
\mathcal{F}_{y_j \to y_i \mid \text{All others } y} = \log \frac{|\Sigma^R_{ii}|}{|\Sigma_{ii}|} \tag{7}
$$

where $|\Sigma_{ii}^R|$ and $|\Sigma_{ii}|$ are prediction error covariance of x_i for reduced model and full model, respectively. Also, both Σ_{ii}^R and Σ_{ii} are calculated from optimal mean-squared error estimation which is Kalman filter (further details in section 3.2). The result of Granger causality test : when $\mathcal{F}_{y_j \to y_i \mid \text{All others } y} > 0$ because $y_i(t)$ in full model usually have more fitting than $y_i(t)$ in reduced model and $\mathcal{F} = 0$ means $\Sigma_{ii}^R = \Sigma_{ii}$. Therefore, $y_j(t)$ does not cause $y_i(t)$.

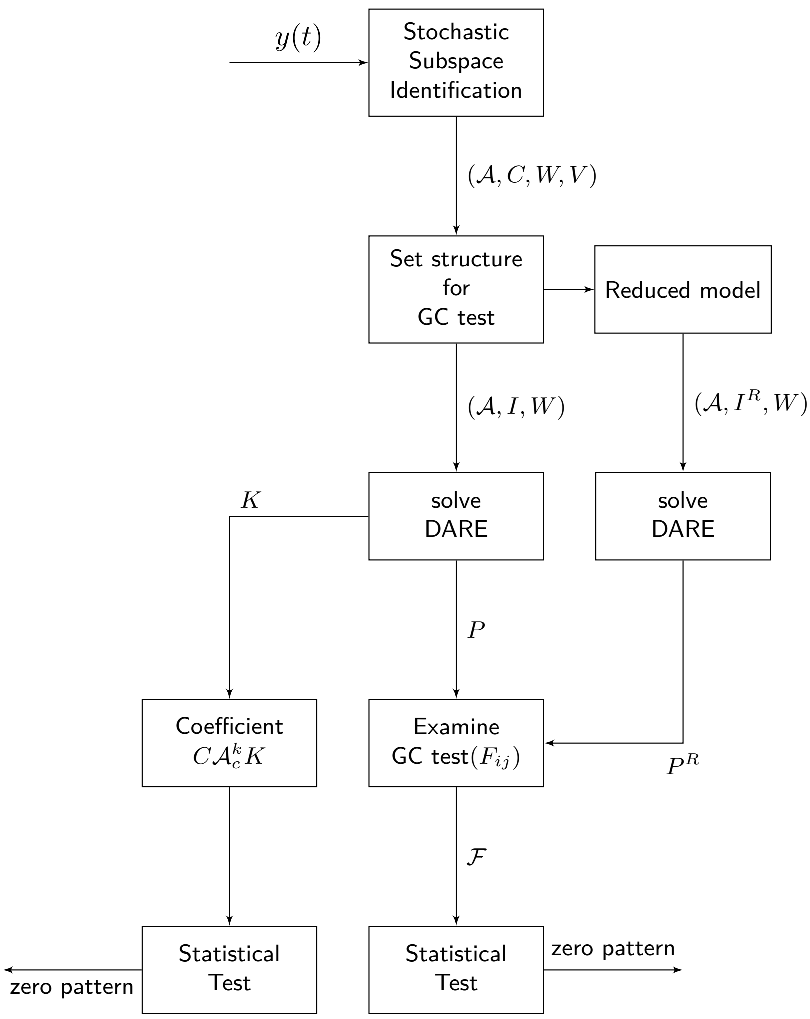
2 Problem Statement

In this project, There are two main objectives by the following.

- **Problem 1** : We estimate the system matrices of state space (6) with free parameters from time-series data by using subspace method because only measurement variable (y) are given and signal source (z) are not measured.
- **Problem 2** : We examine Granger causality test from any estimated state space model which are estimated by using subspace method from the previous problem. The test result from (7) ($\mathcal{F}$) is real source (x). The result from GC test can refer to brain connectivity.

3 Methodology

In this project, we estimate system matrices of state space model without structure from (6). We choose state space model to examine Granger causality. The scheme of model estimation is shown in the following diagram :



Our scheme starts with time-series data $y(t)$ which is the only data we know. $y(t)$ is generated based on Autoregressive ground truth model. From (6), we do not know sources : $x(t)$ and internal noise : $u(t)$ which is problem to compute system matrices since we want to estimate $\mathcal{A}$. No parameters are known. The variables and parameters describe in the following table.

Table 1: Variables and parameters in this project

Variables		Parameters	
Measured	Unknown	Known	Unknown
$y(t)$	$x(t)$		$\mathcal{A}$
	$u(t)$		C
			W, V

We use subspace method to identify all system matrices ($\mathcal{A}$, C and noise covariance). Then, set the estimated structure for Granger causality test by letting $C = I$ for full model and forced j^{th} column of full model C for reduced model, denoted as C^R or I^R for C that remove j^{th} column. There are two method to examine Granger causality test. First, solve discrete Riccati equation for both model that the solution is covariance of prediction error and compare the covariance of prediction of reduced model to full model by using Granger causality test (7), the result ($\mathcal{F}$ have to be verified by statistical test to make sure that the zero pattern of model is satisfy. Another method is to solve gain matrix from DARE (K) and examine coefficient $C\mathcal{A}_c^k K$. The result which verified by statistical test is also zero pattern of model.

3.1 Stochastic subspace method

We estimate sources and system matrices (in this case : $\mathcal{A}$, C,W,V) by using stochastic subspace method. The estimation process starts by estimating sources. Since, EEG linear model have no input so that the estimation will use stochastic subspace method. In this method we focus on estimate state sequence first. The process starts by dividing data by time to obtain past data and future data. Then, project the future output (Y_f) onto the past output (Y_p) space with zero initial state ($\hat{X}_0 = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \end{bmatrix}$) [8].

$$\mathcal{O}_i \triangleq Y_{i\,|\,2i-1}/Y_{0\,|\,i-1} = Y_f/Y_p \tag{8}$$

where $\mathcal{O}_i$ is the oblique projection and $Y_{0\,|\,i-1}$ is measurement data from $t = 0$ to $t = i - 1$. After that, compute the state from single value decomposition (SVD) factorization.

$$\mathcal{O}_i = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_n V_1^T \tag{9}$$

Since $\mathcal{O}_i = \Gamma_i \hat{X}_i$ [9] and there are some non-singular matrix T that $\Gamma_i = U_1 \Sigma_n^{1/2} T$ so that we obtain

$$\hat{X}_i = \Gamma_i^{\dagger} \mathcal{O}_i \tag{10}$$

Then, estimate system matrices in least-square sense by forming the equation

$$\begin{aligned} \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i\,|i} \end{bmatrix} &= \begin{bmatrix} A \\ C \end{bmatrix} \hat{X}_i + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \\ \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} &= \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i\,|\,i} \end{bmatrix} \hat{X}_i^{\dagger} \end{aligned} \tag{11}$$

with noise covariance as

$$\begin{bmatrix} \hat{W} & \hat{S} \\ \hat{S}^T & \hat{V} \end{bmatrix} = (1/j) \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix}^T \tag{12}$$

3.2 Granger causality test on estimated state space model

This process happens after structured system matrices are solved. We use Granger causality test to examine brain connectivity of estimated model. In this process, Before the process starts, we assume that there is no measurement noise and $u(t)$ is also uncorrelated. In this process, we examine Granger causality test in two models : full model and reduced model. For full model, we assume $y(t)$ has the same dimension as $x(t)$. That means we force number of measurement sources equals to number of brain sources. This means we let $C = I$. To reduce the full model, we assume each $C_j = 0$ (column j of C) which means we assume that value x_j does not cause all others x (since we assume that $y(t)_j = x(t)_j$ for all j) . Therefore, $y(t)$ is linear combination of all x except for x_j.

$$\text{Full model :} \qquad z(t+1) = \mathcal{A}z(t) + w(t)\ ,\ y(t) = Cz(t) \tag{13a}$$

$$\text{Reduced model :} \qquad z(t+1) = \mathcal{A}z(t) + w(t)\ ,\ y(t) = C^R z(t) \tag{13b}$$

where $C = I$ and C^R is reduced matrix that the j^{th} column of C is zero.
After that, we find estimation error covariance : $\Sigma = Cov(z - \hat{z}_{t\,|\,t-1})$ of both model. To obtain optimal prediction error covariance, we estimate $\hat{z}$ by using minimum mean square error because with this method, the error from noise is minimized. Therefore, we will get $\hat{x} = \mathbf{E}\{x_t \mid x_{t-1}^-\}$where y_{t-1}^- is all output data from the past up to time $t-1$ and $\hat{x} = \mathbf{E}\{x_t^R \mid x_{t-1}^{R-}\}$ for reduced model. After this, we calculate estimation error covariance by using Kalman Filter [10] because of optimal method in linear model form :

$$\begin{aligned}\hat{x}_{t+1\,|\,t} &= \mathcal{A}\hat{x}_{t\,|\,t-1} + \mathcal{A}\Sigma_{t\,|\,t-1}C^T(C\Sigma_{t\,|\,t-1}C^T + R)^{-1}(y_t - C\hat{x}_{t\,|\,t-1}) \\ &= \mathcal{A}\hat{x}_{t\,|\,t-1} + K(y_t - \hat{y}_{t\,|\,t-1})\end{aligned} \tag{14}$$

where $K = \mathcal{A}\Sigma_{t\,|\,t-1}C^\intercal(C\Sigma_{t\,|\,t-1}C^T + V)^{-1}$ is Kalman gain K from (6) and w_t can be expressed by $y_t - \hat{y}_{t\,|\,t-1}$ where $\hat{y}$ is estimated by MMSE ($\hat{y} = \mathbf{E}\{y_t \mid y_{t-1}^-\}$) and for reduced model we will get ε^R from $y_t^R - \mathbf{E}\{y_t^R \mid y_{t-1}^{R-}\}$. From (14), time update gives a recursive solution. Therefore, we can express measurement and time update of Σ as Riccati recursion [10].

$$\begin{aligned}\Sigma_{t+1\,|\,t} &= \mathcal{A}\Sigma_{t\,|\,t-1}\mathcal{A}^T + W - \mathcal{A}\Sigma_{t\,|\,t-1}C^T(C\Sigma_{t\,|\,t-1}C^T + V)^{-1}C\Sigma_{t\,|\,t-1}\mathcal{A}^T \\ &= \mathcal{A}\Sigma_{t\,|\,t-1}\mathcal{A}^T + W - \mathcal{A}\Sigma_{t\,|\,t-1}C^T(C\Sigma_{t\,|\,t-1}C^T)^{-1}C\Sigma_{t\,|\,t-1}\mathcal{A}^T\end{aligned}$$

(Assume that V is zero)

(15)

From (15) , this equation is the optimal way to find state prediction error covariance [11]. However, we assume observation noise covariance is positive definite, $(\mathcal{A}, C)$ are observable and $(\mathcal{A}, W)$ are controllable so that we can solve steady state Kalman filter instead. The estimation of steady state Kalman filter satisfies Discrete Algebaric Riccati Equation (DARE) :

$$\Sigma = \mathcal{A}\Sigma\mathcal{A}^T + W - \mathcal{A}\Sigma C^T(C\Sigma C^T)^{-1}C\Sigma\mathcal{A}^T \tag{16}$$

There are two methods to examine Granger causality. The first method is to find log ratio of covarience of prediction error (Σ from solving of Riccati equation). Then, we suggest to determine the time-domain Granger causality shown as : [7]

$$\mathcal{F}_{x_j \to x_i\,|\,\text{All others } x} = \log\frac{|\Sigma_{ii}^R|}{|\Sigma_{ii}|} \tag{17}$$

where $|\Sigma_{ii}^R|$ and $|\Sigma_{ii}|$ is estimation error covariance of x_i for reduced model and full model, respectively. In general, Σ_{ii}^R is usually larger than Σ_{ii} because variance is minimized when data is added. If the result is zero, it means $|\Sigma_{ii}^R| = |\Sigma_{ii}|$. Therefore, x_j does not affect x_i conditioning to all others x. Otherwise, the value is always positive because reduced model is come up with more covariance magnitude.

After we solve (16) the solution of DARE remains only W when we assume no measure noise and $u(t)$ is uncorrelated. (See Appendix 6.1) Thus, K from (14) by this assumption shown as :

$$\begin{aligned} K &= \mathcal{A}\Sigma C^T(C\Sigma C^T + V)^{-1} \\ &= \mathcal{A}\begin{bmatrix}\Sigma_{11}^T & \Sigma_{12}^T & \dots & \Sigma_{1p}^T\end{bmatrix}^T \Sigma_{11}^{-1} \\ \text{Because only } \Sigma_{11} \text{ is nonzero matrix} \\ &= \mathcal{A}\begin{bmatrix}I & 0 & \dots & 0\end{bmatrix}^T \\ &= \begin{bmatrix}A_1^T & I & 0 & \dots & 0\end{bmatrix}^T \end{aligned} \tag{18}$$

Another method to examine Granger causality is to find the coefficient $C_i\mathcal{A}_c^k K_j$ when $k = 0, 1, \dots, p-1$. Denote $\mathcal{A}_c$ as state observer closed loop observer gain, the results of coefficient are A_{k+1} for all k which have same structure. Therefore, $\mathcal{A}_c$ yields the necessary and sufficient condition by the Cayley-Hamilton Theorem. (See Appendix 6.2)

4 Experiments

There are two experiments in this section. Both experiments are tested from same information, simultaneously. Also, both experiments are defined from AR ground truth model from (6). The objectives of the first experiment is to compare $\mathcal{F}$ when we solved P by solving DARE from (16) with P by let $P = W$. The result should be the same structure from the calculation in the previous section. The objective of the second experiment is to measure Granger causality test from matrices that estimated by using subspace identification. The result of this experiment should be the same structure as previous experiment since we generate from the same ground truth AR model.
The following example, ground truth AR model is generated with three lags (p) with three observer variables. From Fig. 3, we have structure of $\mathcal{A}$ from `gen_sparseAR.m`.

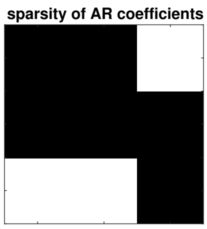


Figure 3: The sparsity for each A_i that generated from `gen_sparseAR.m`

```
>> A

A =

  -0.1095   -0.0665         0    0.0317   -0.1316         0   -0.0013   -0.2247         0
   0.1239    0.2266   -0.1540   -0.2102    0.1622   -0.0998   -0.0735   -0.0335    0.0816
        0         0   -0.3225         0         0    0.0523         0         0    0.0154
   1.0000         0         0         0         0         0         0         0         0
        0    1.0000         0         0         0         0         0         0         0
        0         0    1.0000         0         0         0         0         0         0
```

```
         0         0         0    1.0000         0         0         0         0         0
         0         0         0         0    1.0000         0         0         0         0
         0         0         0         0         0    1.0000         0         0         0
```

4.1 Verify GC test result from DARE compare to reduced DARE

The object of this experiment is to show the equivalence of GC tests on AR model and state space model where ground truth model is AR model. Since we know that the result of GC test from AR model (5) can be derived as $(A_k)_{ij} = 0$ (that means $x_j(t)$ does not cause $x_i(t)$), the expected outcome of GC test on state space model based on ground truth AR model should be the same as the result from AR model. In this experiment, ground truth AR model is generated from MATLAB file : `gen_sparseAR.m` [12]. The process of this experiment starts with format state space model from ground truth AR model.

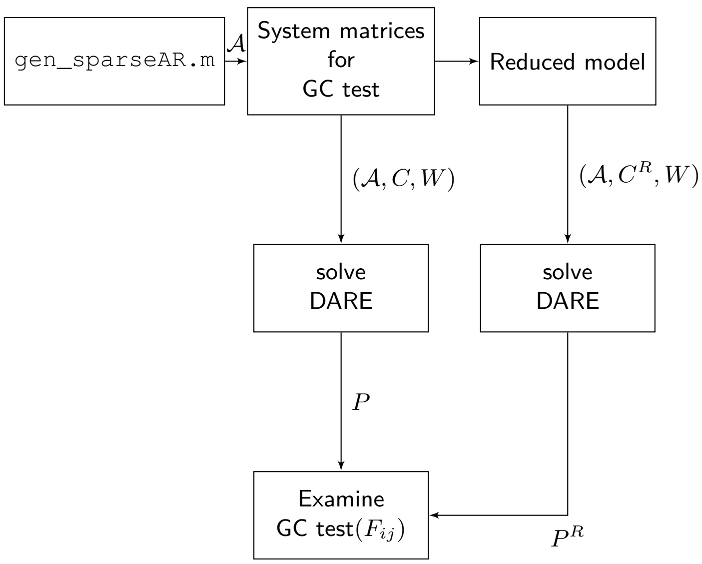
$$z(t+1) = \mathcal{A}z(t) + w(t) \tag{19a}$$

$$y(t) = Cz(t) + \varepsilon(t) \tag{19b}$$

where

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_p \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix}, \ C = \begin{bmatrix} L & 0 & \dots & 0 \end{bmatrix}, \ w(t) = \begin{bmatrix} u(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \varepsilon(t) = \begin{bmatrix} v(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then, we set state space system matrices for GC test by given $y(t) = x(t)$. Therefore, C for GC test in this experiment becomes $C = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}$ for full model. For reduced model, the structure of C^R for reduced model is C for full model which j^{th} column is removed. Moreover, we let measurement noise ($\varepsilon(t)$) to be zero and signal noise are uncorrelated for GC test. After that, we solve Riccati equation for both model so that we obtained residual error for both models. Finally, we examine GC test from (17). The process of this section shown as follow :




```
>> F

F =

    0.0138    0.1019    0.0000
    0.0614    0.0954    0.0423
    0.0000    0.0000    0.1232
```

```
>> F_r

F_r =

    0.0138    0.1019    0.0000
    0.0614    0.0954    0.0423
    0.0000    0.0000    0.1232
```

where $\mathcal{F}$ is function from Granger causality test by solving DARE and $\mathcal{F}_r$ is function from the test that given $\Sigma = W$. The result showed that each element of $\mathcal{F}$ and $\mathcal{F}_r$ is the same value so that we can reduced DARE when there is no measurement noise (V). Another measurement is to measure coefficient $C\mathcal{A}_c^k K$ for $k = 0, 1, \ldots, p-1$.

```
>> CBK

CBK(:,:,1) =

   -0.1095   -0.0665    0.0000
    0.1239    0.2266   -0.1540
   -0.0000   -0.0000   -0.3225


CBK(:,:,2) =

    0.0317   -0.1316    0.0000
   -0.2102    0.1622   -0.0998
    0.0000   -0.0000    0.0523


CBK(:,:,3) =

   -0.0013   -0.2247    0.0000
   -0.0735   -0.0335    0.0816
   -0.0000    0.0000    0.0154
```

`CBK(:,:,i)` is the result of $C\mathcal{A}_c^{i-1}K$. The result satisfied that $C\mathcal{A}_c^k K = A_{k+1}$ which each A_i has the same structure.

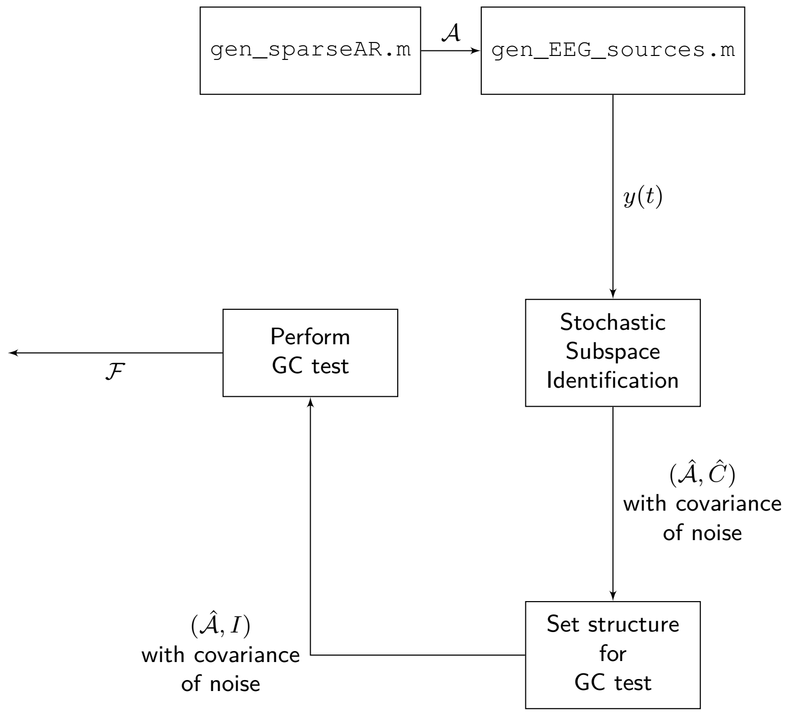
4.2 Subspace Identification

After we verified that GC test from state space models with ground truth AR model give the same result as GC test on AR model, we perform GC test for any estimated state space models in this experiment. The expected result of GC test on estimated state space model in this experiment is the same result as GC test on AR model. The estimated state space models were obtained by using subspace method with time series data from `gen_EEG_sources.m` based on system matrices from (19) which were generated from `gen_sparseAR.m` and lead field matrix L is random with normal distribution. In procedure of subspace method, we use `n4sid.m` which is one of the subspace methods to determine system matrices and also noise variance in terms of innovation form [13] [14].

$$x(t+1) = Ax(t) + Ke(t) \tag{20a}$$

$$y(t) = Cx(t) + e(t) \tag{20b}$$

We assume that the dimension of estimated state space matrices are the same as dimension of all system matrices from (19). After subspace method were done, we obtained all system matrices $(\hat{A}, \hat{C}$ with covariance of noise). Then, we set state space model for GC test (13). Finally, we examine GC test from (17). The process of this section shown as follow :



We compared Granger causality test result between Autoregressive model (F) and estimated state space model (F_ss). For estimated state space model, we choose np order because there is the same dimension as state space model with Autoregressive ground truth model. The results showed that $\mathcal{F}$ from state space model that estimated from subspace identification are not the same value and not the same structure as AR model GC test result
The following command lines are the result of subspace method by using `n4sid`.

```
>> m_free.A

ans =

    0.8404    0.0426    0.3234    0.1713    0.0201    0.1313    0.1585   -0.0846    0.1962
    0.0382   -0.5104    0.4009   -0.2055   -0.8570    0.1781    0.4212   -0.5479   -0.8056
   -0.2885   -0.7280    0.2499   -0.2523   -0.4335   -0.3711   -0.0421   -0.5630    0.3287
    0.0272   -0.0407    0.0019   -0.7313   -0.1302    0.2104   -0.1701    0.2910    0.1178
   -0.0250    0.0103    0.2463    0.3598   -0.8430   -0.2462   -0.3005    0.2350    0.0208
   -0.0961   -0.2028    0.1553    0.1651    0.0630    0.4494   -0.6612    0.1000    0.0288
   -0.2600   -0.0028   -0.2560    0.1142   -0.1336    0.7016    0.4565    0.0333   -0.1161
    0.1145    0.0036    0.0641   -0.0586    0.2121   -0.4329   -0.1152   -0.2433   -0.5690
    0.0023    0.0499   -0.0816   -0.0631   -0.0847   -0.0330    0.4278    0.5835   -0.1160

>> m_free.C

ans =

   -0.1071    0.2170    0.1305   -0.2884   -0.1751    0.0069   -0.0754   -0.2888   -0.0261
   -0.0760    0.1371    0.0841   -0.1886   -0.1337    0.0074   -0.0535   -0.1948   -0.0234
   -0.0617    0.3300    0.1171   -0.4557   -0.2822   -0.0049   -0.1486   -0.4549   -0.0420

>> m_free.K

ans =

    0.3378   -0.3066   -0.1211
   -0.2572    0.2233    0.3220
    0.4736   -0.3590   -0.0253
    0.0741   -0.0272   -0.1319
   -0.3851    0.4753   -0.0151
    0.0074    0.0540   -0.0168
    0.0217   -0.0167    0.0100
   -0.0386    0.1176   -0.1028
```

```
   -0.1682    0.2019    0.0617

>> m_free.NoiseVariance

ans =

   1.0e-03 *

    0.2911    0.1987    0.4106
    0.1987    0.1453    0.2778
    0.4106    0.2778    0.8115
```

After we obtain system matrices, we set $C = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}$, then examine Granger causality test with function `GCTest.m`

```
>> F

F =

    0.0138    0.1019    0.0000
    0.0614    0.0954    0.0423
    0.0000    0.0000    0.1232

>> F_ss

F_ss =

    0.1427    2.2088    0.1189
    0.0390    0.9191    0.1049
    0.9992    1.3230    0.0430

>> CBK_ss

CBK_ss(:,:,1) =

    0.1643   -0.0624   -0.1159
    0.2141   -0.4245    0.3033
    0.5900   -0.5177   -0.0304


CBK_ss(:,:,2) =

    0.4062   -0.3262   -0.0625
    0.0716   -0.0771    0.0636
   -0.1300    0.0751    0.1631


CBK_ss(:,:,3) =

    0.2099   -0.1713    0.0321
   -0.2578    0.1299    0.1008
   -0.6079    0.4997    0.0983
```

The reason can be probably the fitting of estimated model (m_free : $\hat{y}$) is not high enough compared to actual data. The comparison of time series data are shown in Fig. 4
As we look closer at the estimated time series data, the estimated data ($\hat{y}$) followed the actual

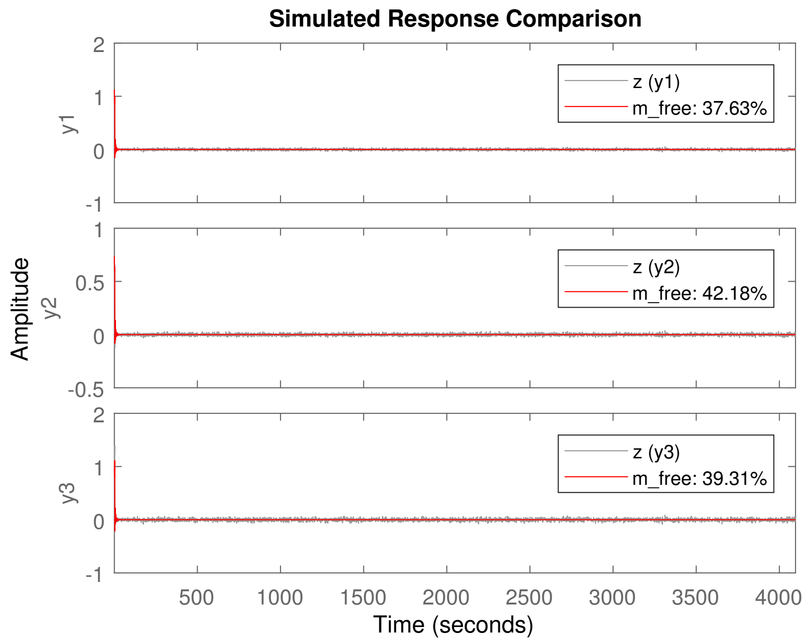


Figure 4: Compare time series data with the data that we estimate from subspace identification with 4097 samples

data (y) until reach 25 datas in which $\hat{y}$ amplitude started to be unchanged follow to y

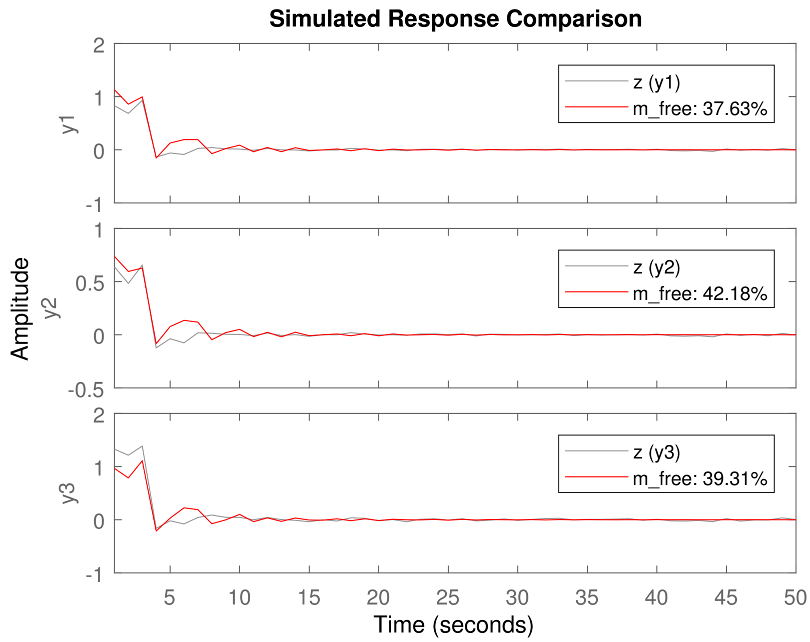


Figure 5: Compare time series data with the data that we estimate from subspace identification at first 50 datas

5 Conclusions

Subspace identification is the method used for estimate system matrices and state sequence. This method can be applied in EEG time series data that could be described in Autoregressive (AR) model. We learn brain connectivity of state space model by using Granger causality (GC) test. We performed GC test and compare the result. It showed that GC test from AR model come up with the same result as GC test from state space model. Therefore, we can apply EEG time series data into state space model. Moreover, we examine GC test based on state space model that are estimated from time series data with ground truth AR model by using subspace method. The result of that GC test is not the same structure as the structure of AR model. From the fitting of estimation sources ($\hat{y}$), we imply that estimation error is one of causes for failure experiments. However, we need to explore more cause in this experiments and learn about solution leads to expected result.

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6 Appendices

6.1 Simplification of DARE applied to AR model

For AR model case, we examine GC test that $x_j(t)$ causes or does not cause $x_i(t)$ by comparing noise covariance of reduced model (Σ_{ii}^R : Noise covariance when we remove x_j from the model.) and full model by (5). When we determine GC test on state space model based on AR model (19), noise covariance can be calculated by using steady state Kalman filter that satisfies discrete Riccati equation (16). In this section, we demonstrate that the solution of discrete Riccati equation can be simplified to $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}$ We can simplify Σ that we solve from into more similar form (16) by given $\left(W \in \mathbf{R}^{np\times np} \text{ and } W = \begin{bmatrix} W_1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix} \right)$

Given $\mathbb{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{p-1} \end{bmatrix}$ and $\Sigma \in \mathbf{R}^{np\times np}$ are in formation as : $\left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right]$ where $U_{ij} \in \mathbf{R}^{p\times p}$ denote the $(i,j)^{th}$ block of U

From

$$\Sigma = \mathcal{A}\Sigma\mathcal{A}^T + W - \mathcal{A}\Sigma C^T(C\Sigma C^T)^{-1}C\Sigma\mathcal{A}^T$$

we simplify in each term

$$\begin{aligned} \mathcal{A}\Sigma\mathcal{A}^T &= \left[\begin{array}{c|c} \mathbb{A} & A_p \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right] \left[\begin{array}{c|c} \mathbb{A}^T & I \\ \hline A_p^T & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbb{A}U\mathbb{A}^T + A_pV^T\mathbb{A}^T + \mathbb{A}VA_p^T + A_pRA_p^T & \mathbb{A}U + A_pV^T \\ \hline U\mathbb{A}^T + VA_p^T & U \\ \vdots & \end{array}\right] \end{aligned} \tag{21}$$

$$\begin{aligned} \mathcal{A}\Sigma C^T &= \left[\begin{array}{c|c} \mathbb{A} & A_p \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} U & V \\ \hline V^T & U \end{array}\right] \left[\begin{array}{c} I \\ \hline 0 \\ \\ 0 \end{array}\right] \\ &= \text{First block column of } \left[\begin{array}{c} \mathbb{A}U + A_pV^T \\ \hline U \end{array}\right] \\ &= \left[\begin{array}{c} \sum_{i=1}^{p} A_i\Sigma_{i,1} \\ \hline \Sigma_{11} \\ \Sigma_{21} \\ \\ \Sigma_{p-1,1} \end{array}\right] \end{aligned} \tag{22}$$

$$C\Sigma\mathcal{A}^T = (\mathcal{A}\Sigma C^T)^T = \left[\begin{array}{c|cccc} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T & \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \end{array}\right] \tag{23}$$

$$C\Sigma C^T = \Sigma_{11} \tag{24}$$

Then, combine all above terms (21), (22), (23) and (24) into DARE

$$\left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}U\mathbb{A}^T + A_pV^T\mathbb{A}^T + \mathbb{A}VA_p^T + A_pRA_p^T & \mathbb{A}U + A_pV^T \\ \hline U\mathbb{A}^T + VA_p^T & U \end{array}\right]$$
$$+ \begin{bmatrix} W_1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix} - \left[\begin{array}{c} \sum_{i=1}^{p} A_i\Sigma_{i,1} \\ \hline \Sigma_{11} \\ \Sigma_{21} \\ \vdots \\ \Sigma_{p-1,1} \end{array}\right] (\Sigma_{11})^{-1} \left[\begin{array}{c|cccc} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T & \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \end{array}\right] \tag{25}$$

From (21)

$$\begin{aligned} U\mathbb{A}^T + VA_p^T &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \\ \Sigma_{21} & \ddots & & \Sigma_{2,p-1} \\ \vdots & & \ddots & \vdots \\ \Sigma_{p-1,1} & & & \Sigma_{p-1,p-1} \end{bmatrix} \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_{p-1}^T \end{bmatrix} + \begin{bmatrix} \Sigma_{1,p} \\ \Sigma_{2,p} \\ \vdots \\ \Sigma_{p-1,p} \end{bmatrix} A_p^T \\ &= \begin{bmatrix} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\ \sum_{i=1}^{p} \Sigma_{2,i}A_i^T \\ \vdots \\ \sum_{i=1}^{p} \Sigma_{p-1,i}A_i^T \end{bmatrix} \end{aligned} \tag{26}$$

$$\text{Determine } \Sigma_{21} \qquad \begin{aligned} \Sigma_{21} &= (\text{First row blocks of } U\mathbb{A}^T + VA_p^T) - \Sigma_{11}(\Sigma_{11}^{-1}) \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\ &= \sum_{i=1}^{p} \Sigma_{1,i}A_i^T - \sum_{i=1}^{p} \Sigma_{1,i}A_i^T = 0 \end{aligned}$$

For the others i,$\Sigma_{2,i} = \Sigma_{1,i} - \Sigma_{11}(\Sigma_{1,1})^{-1}\Sigma_{1,i} = 0$. This means $\Sigma_{2,i} = 0$ for all $i = 1, 2, \dots, p$

$$\text{Determine } \Sigma_{31} \qquad \begin{aligned} \Sigma_{31} &= (\text{Second row blocks of } U\mathbb{A}^T + VA_p^T) - \Sigma_{21}(\Sigma_{11}^{-1}) \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\ &= \sum_{i=1}^{p} \Sigma_{2,i}A_i^T = 0 \ (\Sigma_{2,i} = 0 \text{ for all } i) \end{aligned}$$

For the others i,$\Sigma_{3,i} = \Sigma_{2,i} - \Sigma_{21}(\Sigma_{11})^{-1}\Sigma_{2,i} = 0$ where $i = 2, 3, \dots, p$. This means $\Sigma_{3,i} = 0$ for all $i = 1, 2, \dots, p$
Consequently, $\Sigma_{i,j} = 0$ for all $i = 1, 2, \dots, p, j = 1, 2, \dots, p$ except for Σ_{11}

$$\text{Determine } \Sigma_{11} \qquad \begin{aligned} \Sigma_{11} &= A_1\Sigma_{11}A_1^T + W_1 - A_1\Sigma_{11}(\Sigma_{11})^{-1}\Sigma_{1,1}A_1^T \\ &= W_1 \end{aligned}$$

The result of Riccati equation remains only block $\Sigma_{11} = W_1$. Therefore, it satisfies that $\Sigma = W$

6.2 $C\mathcal{A}_c^k K$ coefficients

The results of GC test on AR model (5) can be derived as coefficient $(A_k)_{ij} = 0, \forall k$ which means $x_j(t)$ does not cause $x_i(t)$. Meanwhile, the results of GC test on state space model (7) can also be measured by $C\mathcal{A}_c^k K$ coefficient. When $C_i\mathcal{A}^k K_j = 0, \forall k,\ i = 1, \dots, n-1$, it means $x_j(t)$ does not cause $x_i(t)$ In this section, we showed that the coefficient from GC test on AR model have same structure to the coefficient from GC test on state space based on ground truth AR model.

$$\text{Coefficient of GC test on AR model :} \quad (A_k)_{ij} = 0\,, \forall k \tag{27a}$$

$$\text{Coefficient of GC test on state space model :} \quad C_i\mathcal{A}^k K_j = 0\,, \forall k,\ i = 1, \dots, n-1 \tag{27b}$$

Given state observer closed loop observer gain $\mathcal{A}_c = \mathcal{A} - KC$. From (6) we have

$$\begin{aligned}\mathcal{A}_c &= \mathcal{A} - KC \\ &= \begin{bmatrix} 0 & A_2 & \dots & A_p \\ 0 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix}\end{aligned} \tag{28}$$

Then, multiply by C on the left hand side and K on the right hand side :

$$C\mathcal{A}_c^k K = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & A_2 & \dots & A_p \\ 0 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix}^k \begin{bmatrix} A_1 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{29}$$

$$\begin{array}{lll} \text{when } k = 0 & CK & = A_1 \\ \text{when } k = 1 & C\mathcal{A}_c K & = A_2 \\ \text{when } k = 2 & C\mathcal{A}_c^2 K & = A_3 \\ & \vdots & \vdots \\ \text{when } k = p-1 & C\mathcal{A}_c^{p-1} K & = A_p \end{array}$$

Because A_1 , A_2 , ... , A_p have the same structure so that if we assume $(A_1)_{ij} = 0$ that means $(\mathcal{A}_c)_{12} = 0$. Therefore, $\mathcal{A}_c = \mathcal{A} - KC$ yields the necessary and sufficient condition $C_i\mathcal{A}_c^k K_j = 0$ by the Cayley-Hamilton Theorem.

6.3 MATLAB Code

Here is MATLAB code for experiment from section 4 : `main.m` that contain subspace method and GC test

```
clc; clear all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% DATA PARAMETERS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

n           = 3;        % state space and observation variable dimension
p           = 3;        % AR order
density     = 0.5;      % density of AR matrix

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% DATA GENERATION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

 [ind_z,ind_nz,Atrue,ind_z_3D] = gen_sparseAR(n,p,density); %generate AR model
A = [];
for i=1:p
    A = [A Atrue(:,:,i)];
end
A = [A; eye(n*(p-1)), zeros(n*(p-1),n)];
```

```
C = [eye(n) zeros(n,n*(p-1))];    % generate C = [I, 0,...,0 ]
W = zeros(n*p,n*p);
V = zeros(n,n);
E = eye(n*p); % generate V
S = zeros(n*p,n);          % generate S
for i= 1:n
    W(i,i) = rand();
     V(i,i) = 0.00001*rand();
end
%%%%%%%%%%%%% SUBSPACE IDENTIFICATION with GC TEST %%%%%%%%%%%%%
[y,L_true] = gen_EEG_sources(Atrue,n,p);
u = zeros(size(y));
z = iddata(y',u',1);
[m_free,x0] = n4sid(z,n*p,'ssp','free','ts',1); % Choose order np
C_hatGC = zeros(n,n*p);
C_hatGC(1:n,1:n) = eye(n); %Set C_hat for GC Test
W_hat = m_free.K*m_free.NoiseVariance*m_free.K'; % Ke(t)e(t)'K' in n4sid
S_hat = m_free.K*m_free.NoiseVariance; %Ke(t)e(t)' in n4sid
V_hat = m_free.NoiseVariance; %e(t) in n4sid
[F_ss,CBK_ss] = GCTest(m_free.A,C_hatGC,W_hat,V,S,E,n,p);
%%%%%%%%%%%%%%%%%%%% GC TEST for GIVEN STRUCTURE %%%%%%%%%%%%%%%%%%%
[F,CBK] = GCTest(A,C,W,V,S,E,n,p);
F_r = GCReduced(A,C,W,V,n,p); %GC Test when P=W
```

We generate $\mathcal{A}$ from `gen_sparseAR.m`

```
function [ind_z,ind_nz,A,ind_z_3D] = gen_sparseAR(n,p,density)
% gen_sparseARX generates a sparse vector autoregressive model with exogenous
    inputs
% [ind_zz,ind_nz,A,B] = gen_sparseARX(n,p,m,q,noise_var,density,Num)
% This code is generate only p in ARX Model
% ARX
% y(t) = A1*y(t-1) + A2*y(t-2) + ... + Ap*y(t-p)+ B1*u(t-1) + B2*u(t-2) + ... +
    Bq*u(t-q) + e(t)
%
% 'A' represents AR coefficients A1,A2,...,Ap and is stored as a p-dimensional
    array
% 'B' represents X coefficients B1,B2,...,Bq and is stored as a q-dimensional
    array
% The input arguments are
% 'n': dimension of output
% 'p': order of 'AR' in ARX model
% 'm': dimension of input
% 'q': order of 'X' in ARX model
% 'noise_var': variance of u(t) (noise)
% 'density': the fraction of nonzero entries in AR coefficients
% 'Num': number of data points in time series
%
% The AR coefficients are sparse with a common sparsity pattern. The
% indices of nonzero entries are saved in 'ind_nz'.
%
% 'y' is a time series generated from the modAel and has size n x Num
%  y = [y(1) y(2) ... y(Num)]
%
% if p = 0, 'y' is simply a random variable. In this case, A is the
% covariance matrix of u with sparse inverse.

%% Static case
if (p==0),
    S = sparse(2*eye(n)+sign(sprandsym(n,density)));
    [i,j]=find(S);
    S = S+sparse(ceil(max(0,-min(eig(S))))*eye(n));
    A = S\eye(n); % covariance matrix with sparse inverse
    R = chol(phi);
    y = R'*randn(n,Num); % y reduces to a random variable with covariance 'phi'
```

```
    ind_nz = sub2ind([n n],i,j);he
    figure;plot_spy(ind_nz,n,'image');
    title('correct sparsity');
    return;
end

%% Randomize AR coefficients
MAX_EIG = 1;
diag_ind = find(eye(n));
k = length(diag_ind);
diag_ind3D = kron(n^2*(0:p-1)',ones(k,1))+kron(ones(p,1),diag_ind);

A = zeros(n,n,p);
S = sprand(n,n,density)+eye(n);
ii = 0;
while MAX_EIG,
    ii = ii+1;
    for k=1:p,
        A(:,:,k) = 0.1*sprandn(S);
    end
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    poles = -0.7+2*0.7*rand(n,p); % make the poles inside the unit circle
    characeq = zeros(n,p+1);
    for jj=1:n,
        characeq(jj,:) = poly(poles(jj,:)); % each row is [1 -a1 -a2 ... -ap]
    end
    aux = -characeq(:,2:end);
    A(diag_ind3D) = aux(:); % replace the diagonal entries with stable
    polynomial

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    AA =[];
    for k=1:p,
        AA = [AA A(:,:,k)];
    end
    AA = sparse([AA ; [eye(n*(p-1)) zeros(n*(p-1),n)]]);
    if max(abs(eigs(AA))) < 1
        MAX_EIG = 0;
    end
end
abs(eigs(AA))
ii

%% the sparsity pattern of A1,A2,...,Ap
ind_nz = find(S)
ind_z = find(~S);
ind_z_3D = find(~A);
figure;
subplot(1,2,1);
spy(ind_nz,'r',n);title('sparsity of AR coefficients');

%% Generate time series
% noise_var = 0;
%noise = sqrt(noise_var)*randn(n,Num);
%u=rand(m,Num);
%for i=1:Num
    %norm_u=norm(u(:,i));
    %if norm_u <=1/3
    %    u(:,i)=[0;0];
    %elseif norm_u <=1/3
    %    u(:,i)=[1;0];
  %  else
   %    u(:,i)=[0;1];
 %  end
```

```
%end
```

After we obtain $\mathcal{A}$, we can generate time series data from `gen_EEG_sources.m`

```
function [y,L_true] = gen_EEG_sources(A_true,n,p)
 %number of EEG
noise_var=0.01;
density=0.5;

x1p=rand(n,p);
[x]=gen_time_series(A_true,noise_var,x1p);

L_true = randn(n,n);
y = L_true*x;
```

For state space Granger causality test, we usually test from `GCTest.m`, which compute both $\mathcal{F}$ and coefficient of $C\mathcal{A}_c^k K$

```
function [F,CBK] = GCTest(A,C,W,V,S,E,n,p)
%%%%%%%%%%%%%%%%%%%%%%%% Full model %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[P,L,K] = dare(A',C',W,V,S,E);% solve RICCATI

%%%%%%%%%%%%%%%%%%%%%%%% Reduced model %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%---------- solve RICCATI for all reduced model ------------------------
for i=1:n
    Creduce = C;
    Creduce(:,i) = [0]; % force ith column of C to be zero
    [Preduce,L,Kreduce] = dare(A',Creduce',W,V,S,E);
    eig(Preduce)
    CR(:,:,i) = Creduce;
    PR(:,:,i) = Preduce;
    KR(:,:,i) = Kreduce;
end
%--------- Collect all P(i,i) for all reduced model -----------------
diagPR = [];          % P(i,i) for reduced model
for i=1:n
    diagPR = [diagPR diag(PR(:,:,i))];
end
diagPR = diagPR'
diagP = diag(P)'     % P(i,i) for full model
%------------------------------------------------------------------------

%%%%%%%%%%%%%%%%%%%%%%%% Granger Causality %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%------------------ Calculate GC for all components ------------------
F = []; % GC(i,j) is Granger cause from i to j
for i=1:n
    for j=1:n
        F(j,i) = log((diagPR(i,j))/(diagP(j))); %F from covariance matrix
        end
end
for i = 1:p
    CBK(:,:,i) = C*(A-K'*C)^(i-1)*K';
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```