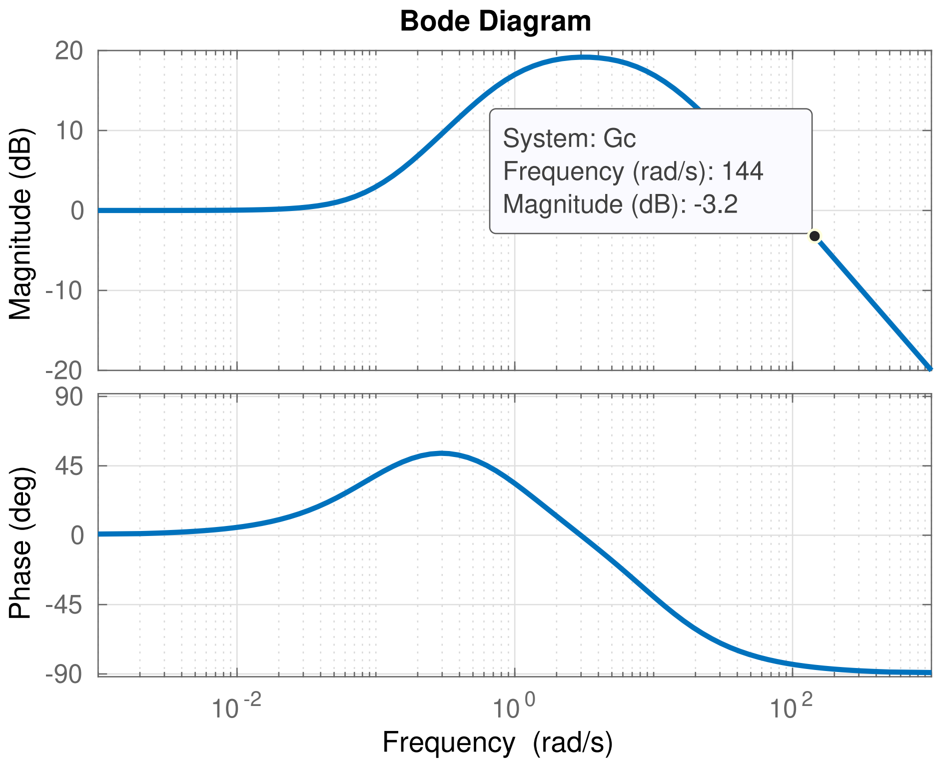
3. Input signals

- common input signals in system identification
  - step function
  - sum of sinusoids
  - ARMA sequences
  - pseudo random binary sequence (PRBS)
- persistent excitation

System description

given a system with the frequency response

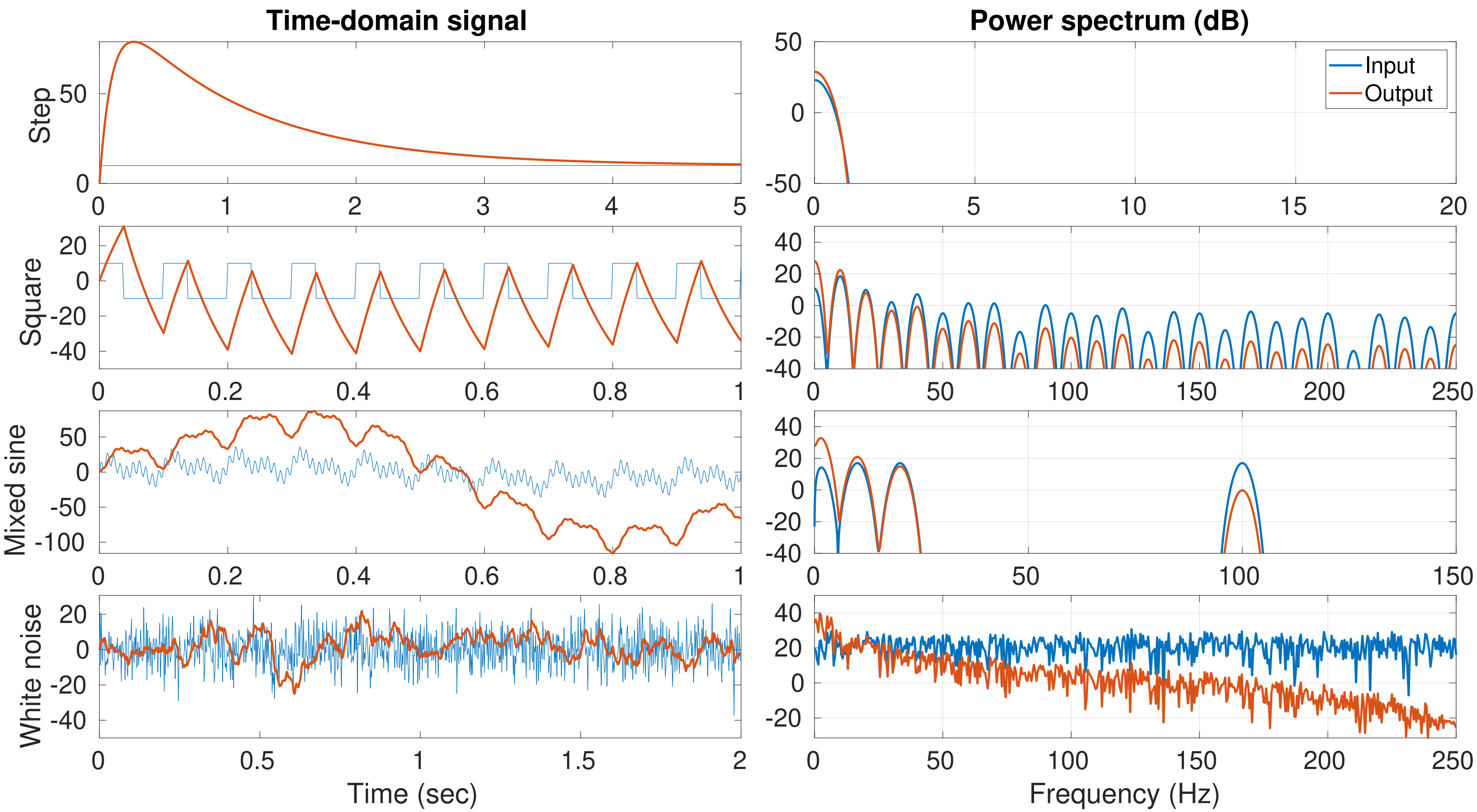


the bandwidth (BW) is around 144 rad/s

the system does not respond to input containing higher frequency than BW

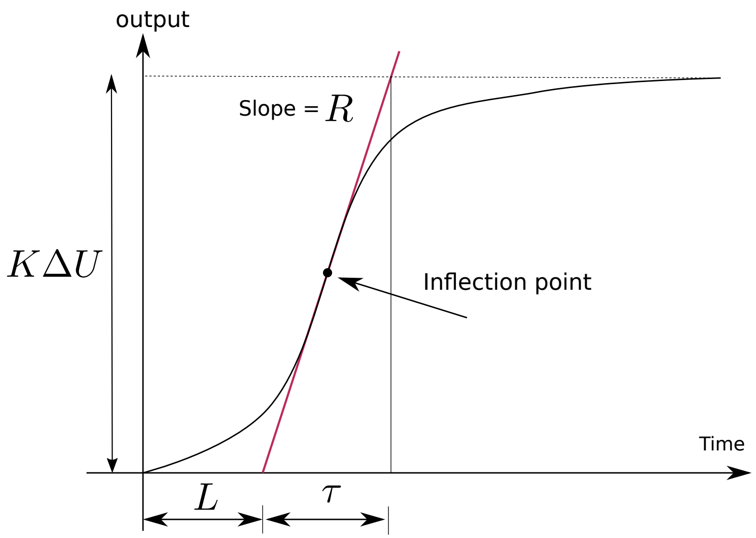
System responses to various inputs

the system responds to inputs differently



the responses to step and mixed sine contain a limited no. of frequency components

Step function



a step function: $u(t) = \Delta U$ for $t \geq 0$

a typical step response of a process

- a step response can be related to rise time, overshoots, static gain
- useful for systems with a large signal-to-noise ratio
- for a simple first-order-plus-time-delay model

$$G(s) = \frac{Ke^{-Ls}}{\tau s + 1}$$

one can consider the reaction curve to estimate K, τ and L from the response

Sum of sinusoids

the input signal $u(t)$ is given by

$$u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$$

where the angular frequencies $\{\omega_k\}$ are distinct,

$$0 \leq \omega_1 < \omega_2 < \ldots < \omega_m \leq \pi$$

and the amplitudes and phases a_k, ϕ_k are chosen by the user

Characterization of sinusoids

let S_N be the average of a sinusoid over N points

$$S_N = \frac{1}{N}\sum_{t=1}^{N} a\sin(\omega t + \phi)$$

Let μ be the mean of the sinusoidal function

$$\mu = \lim_{N\to\infty} S_N = \begin{cases} a\sin\phi, & \omega = 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- $u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$ has zero mean if $\omega_1 > 0$
- WLOG, assume zero mean for $u(t)$ (we can always subtract the mean)

Spectrum of sinusoidal inputs

the autocorrelation function can be computed by

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t+\tau)u(t) = \sum_{k=1}^{m} C_k \cos(\omega_k \tau)$$

with $C_k = a_k^2/2$ for $k = 1, 2, \ldots, m$

if $\omega_m = \pi$, the coefficient C_m is modified to

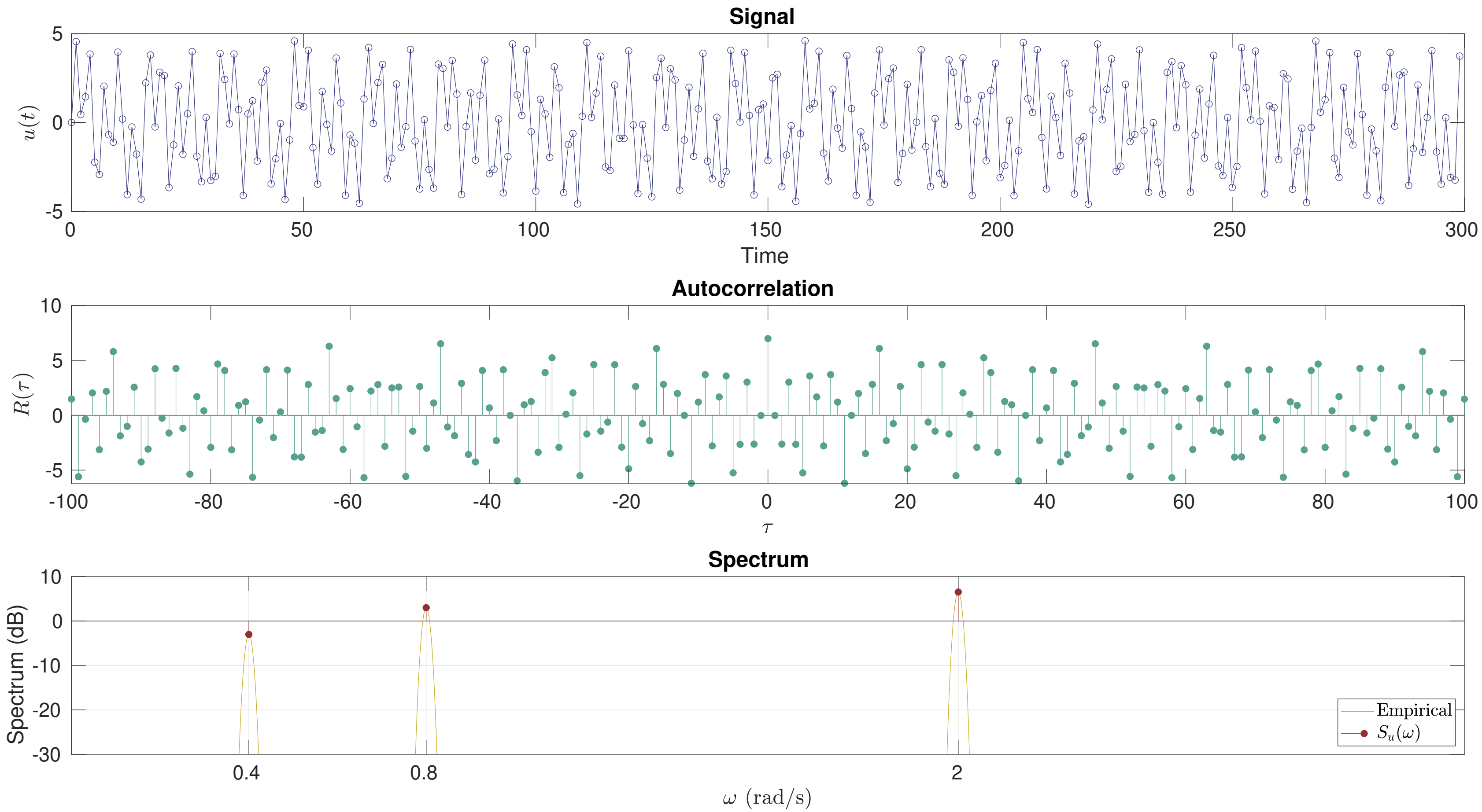
$$C_m = a_m^2 \sin^2(\phi_m)$$

therefore, the spectrum is

$$S(\omega) = \sum_{k=1}^{m} (C_k/2) \left[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)\right], \quad -\pi < \omega \leq \pi$$

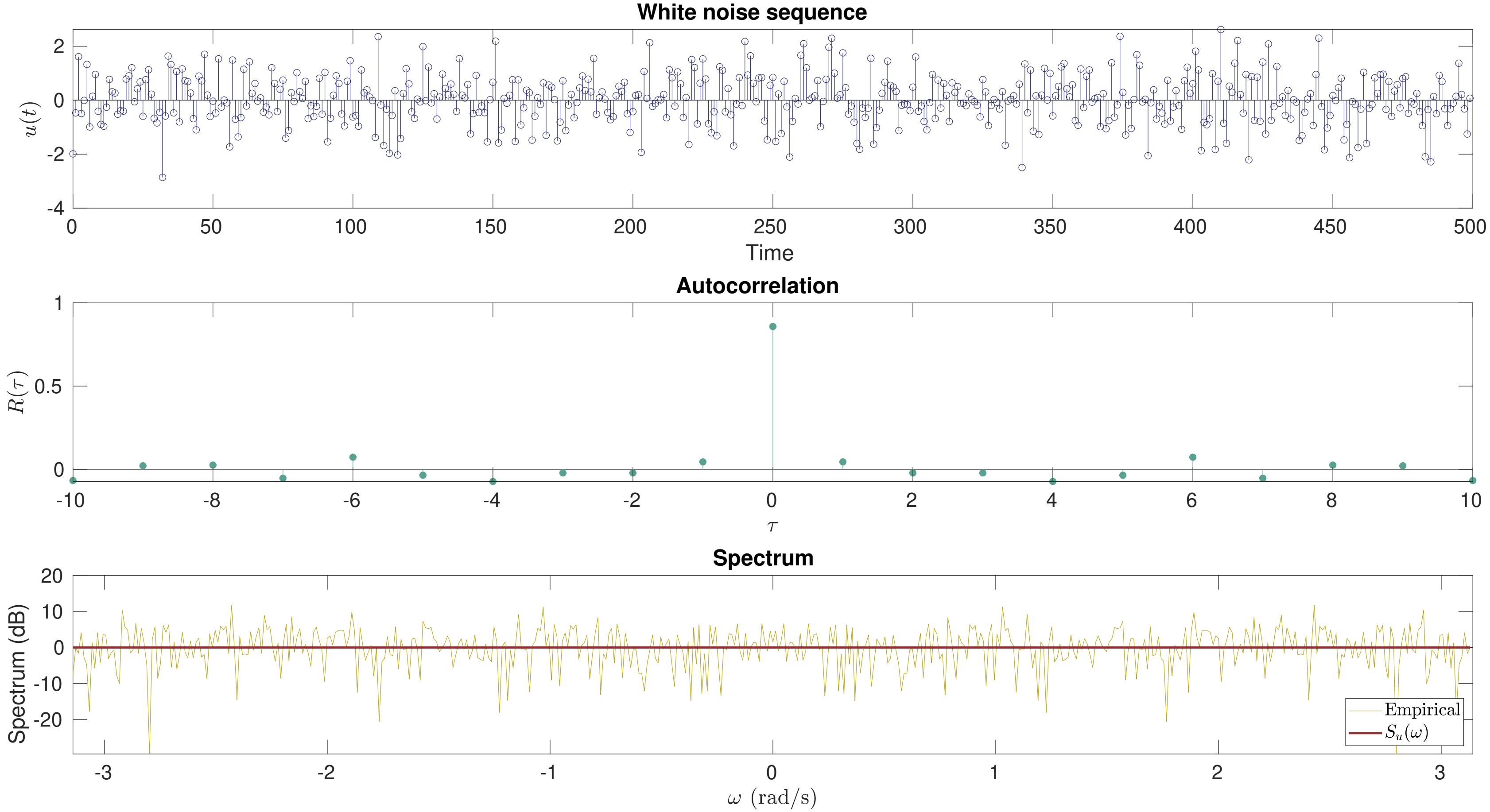
autocorrelation and spectrum of sum of sinusoids

$$u(t) = \sin(0.4t) + 2\sin(0.8t)2\sin(2t)$$



White noise

a white noise input has zero mean and $\mathbf{E}[u(t)u(s)^T] = 0$ for $t \neq s$



a white noise has autocorrelation as delta function and has a flat spectrum

Autoregressive moving average sequence

let $e(t)$ be a pseudorandom sequence similar to white noise in the sense that

$$\frac{1}{N}\sum_{t=1}^{N} e(t)e(t+\tau) \to 0, \quad \text{as } N \to \infty$$

a general input $u(t)$ can be obtained by linear filtering

$$u(t) + c_1 u(t-1) + \cdots + c_p u(t-p) = e(t) + d_1 e(t-1) + \cdots + d_q e(t-q)$$

- u(t) is called *ARMA (autoregressive moving average)* process
- when all $c_i = 0$ it is called *MA (moving average)* process
- when all $d_i = 0$ it is called *AR (autoregressive)* process
- the user gets to choose c_i, d_i and the random generator of $e(t)$

the transfer function from $e(t)$ to $u(t)$ is

$$U(z) = \frac{D(z)}{C(z)} E(z)$$

where

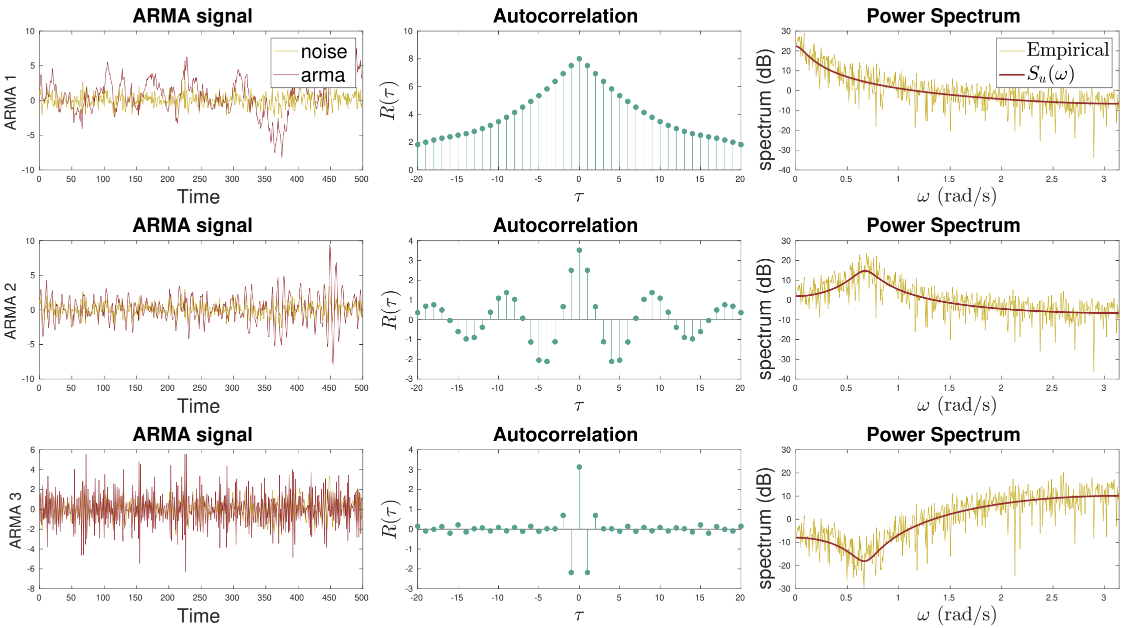
$$C(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots c_p z^{-p}$$
$$D(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \cdots d_q z^{-q}$$

- the distribution of $e(t)$ is often chosen to be Gaussian
- c_i, d_i are chosen such that $C(z), D(z)$ have zeros outside the unit circle
- different choices of c_i, d_i lead to inputs with various spectral characteristics

Spectrum of ARMA process

let $e(t)$ be a white noise with unit variance

$$G_1(z) = \frac{1 + 0.3z^{-1}}{1 - 0.7z^{-1} - 0.2z^{-2}}, G_2(z) = \frac{1 - 0.5z^{-1}}{1 - 1.4z^{-1} + 0.8z^{-2}}, G_3(z) = 1 - 1.4z^{-1} + 0.8z^{-2}$$



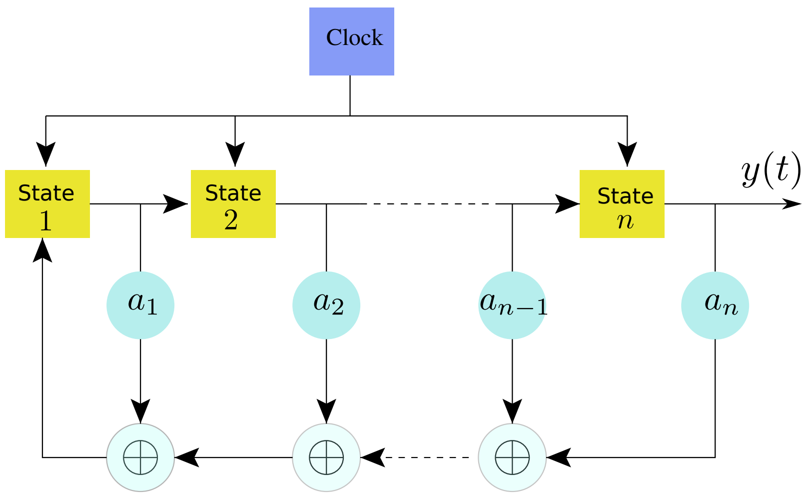
the poles and zeros of G_i explain the waveform of autocorrelation and spectrum

Pseudo Random Binary Sequence (PRBS)

PRBS is generated from a vector autoregressive equation

$$x(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} x(t)$$



$$\begin{matrix} a & x(t) & x(t+1) \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{matrix}$$

$$a_1x_1(t) \oplus a_2x_2(t) \oplus a_3x_3(t) = 0 \oplus 0 \oplus 1 = 1$$

Characteristics of PRBS

- every initial state is allowed except the all-zero states
- the feedback coefficients $a_1, a_2, \ldots, a_n$ are either 0 or 1
- all additions are modulo-two operations (XOR)
- the sequences are two-state signals (binary)
- there are possible $2^n - 1$ different state vectors x (all-zero state is excluded)
- a PRBS of period equal to $M = 2^n - 1$ is called **a maximum length PRBS** (ML PRBS)
- for *maximum length PRBS*, its characteristic resembles white random noise (pseudorandom)

Influence of the Feedback Path

let $n = 3$ and initialize x with $x(0) = (1, 0, 0)$

- with $a = (1, 1, 0)$, the state vectors $x(k), k = 1, 2, \ldots$ are

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix}$$

  the sequence has period equal to 3

- with $a = (1, 0, 1)$, the state vectors $x(k), k = 1, 2, \ldots$ are

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}0\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix}$$

  the sequence has period equal to 7 (the maximum period, $2^3 - 1$)

Maximum length PRBS

denote q^{-1} the unit delay operator and let

$$A(q^{-1}) = 1 \oplus a_1 q^{-1} \oplus a_2 q^{-2} \oplus \cdots \oplus a_n q^{-n}$$

the PRBS $y(t)$ satisfies the homogeneous equation:

$$A(q^{-1})y(t) = 0$$

this equation has only solutions of period $M = 2^n - 1$ *if and only if*

1. the binary polynomial $A(q^{-1})$ is irreducible, *i.e.*, there do not exist any two polynomials $A_1(q^{-1})$ and $A_2(q^{-1})$ such that

$$A(q^{-1}) = A_1(q^{-1})A_2(q^{-1})$$

2. $A(q^{-1})$ is a factor of $1 \oplus q^{-M}$ but is not a factor of $1 \oplus q^{-p}$ for any $p < M$

Generating Maximum length PRBS

examples of polynomials $A(z)$ satisfying the previous two conditions on page 3-16

n	$A(z)$	
3	$1 \oplus z \oplus z^3$	$1 \oplus z^2 \oplus z^3$
4	$1 \oplus z \oplus z^4$	$1 \oplus z^3 \oplus z^4$
5	$1 \oplus z^2 \oplus z^5$	$1 \oplus z^3 \oplus z^5$
6	$1 \oplus z \oplus z^6$	$1 \oplus z^5 \oplus z^6$
7	$1 \oplus z \oplus z^7$	$1 \oplus z^3 \oplus z^7$
8	$1 \oplus z \oplus z^2 \oplus z^7 \oplus z^8$	$1 \oplus z \oplus z^6 \oplus z^7 \oplus z^8$
9	$1 \oplus z^4 \oplus z^9$	$1 \oplus z^5 \oplus z^9$
10	$1 \oplus z^3 \oplus z^{10}$	$1 \oplus z^7 \oplus z^{10}$

Properties of maximum length PRBS

let $y(t)$ be an ML PRBS of period $M = 2^n - 1$

- within one period $y(t)$ contains $(M+1)/2 = 2^{n-1}$ ones and $(M-1)/2 = 2^{n-1} - 1$ zeros
- For $k = 1, 2, \ldots, M-1$,

$$y(t) \oplus y(t-k) = y(t-l)$$

  for some $l \in [1, M-1]$ that depends on k

moreover, for any binary variables x, y,

$$xy = \frac{1}{2}(x + y - (x \oplus y))$$

these properties are used to compute the covariance function of maximum length PRBS

Covariance function of maximum length PRBS

the mean is given by counting the number of outcome 1 in $y(t)$:

$$m = \frac{1}{M}\sum_{t=1}^{M} y(t) = \frac{1}{M}\left(\frac{M+1}{2}\right) = \frac{1}{2} + \frac{1}{2M}$$

the mean is slightly greater than 0.5

using $y^2(t) = y(t)$, we have the covariance function at lag zero as

$$C(0) = \frac{1}{M}\sum_{t=1}^{M} y^2(t) - m^2 = m - m^2 = \frac{M^2-1}{4M^2}$$

the variance is therefore slightly less than $1/4$

Covariance function of maximum length PRBS

for $\tau = 1, 2, \ldots,$

$$\begin{aligned}
C(\tau) &= (1/M) \sum_{t=1}^{M} y(t+\tau) y(t) - m^2 \\
&= \frac{1}{2M} \sum_{t=1}^{M} [y(t+\tau) + y(t) - (y(t+\tau) \oplus y(t))] - m^2 \\
&= m - \frac{1}{2M} \sum_{t=1}^{M} y(t+\tau-l) - m^2 = m/2 - m^2 \\
&= -\frac{M+1}{4M^2}
\end{aligned}$$

Asymptotic behavior of the covariance function of PRBS

Define $\tilde{y}(t) = -1 + 2y(t)$ so that its outcome is either -1 or 1

if M is large enough,

$$\begin{aligned}\tilde{m} &= -1 + 2m = 1/M \approx 0 \\ \tilde{C}(0) &= 4C(0) = 1 - 1/M^2 \approx 1 \\ \tilde{C}(\tau) &= 4C(\tau) = -1/M - 1/M^2 \approx -1/M, \quad \tau = 1, 2, \ldots, M-1\end{aligned}$$

with a large period length M

- the covariance function of PRBS has similar properties to a white noise
- however, their spectral density matrices can be drastically different

Spectral density of PRBS

the output of PRBS sequence is shifted to values $-a$ and a with period M

the autocorrelation function is also periodic and given by

$$R(\tau) = \begin{cases} a^2, & \tau = 0, \pm M, \pm 2M, \dots \\ -\frac{a^2}{M}, & \text{otherwise} \end{cases}$$

since $R(\tau)$ is periodic with period M, it has a Fourier representation:

$$R(\tau) = \sum_{k=0}^{M-1} C_k e^{\mathrm{i}2\pi\tau k/M}, \quad \text{with Fourier coefficients } C_k$$

therefore, the spectrum of PRBS is an impulse train:

$$S(\omega) = \sum_{k=0}^{M-1} C_k \delta\left(\omega - \frac{2\pi k}{M}\right)$$

Spectral density of PRBS

hence, the Fourier coefficients

$$C_k = \frac{1}{M} \sum_{\tau=0}^{M-1} R(\tau) e^{-\mathrm{i}2\pi\tau k/M}$$

are also the spectral coefficients of $S(\omega)$

using the expression of $R(\tau)$, we have

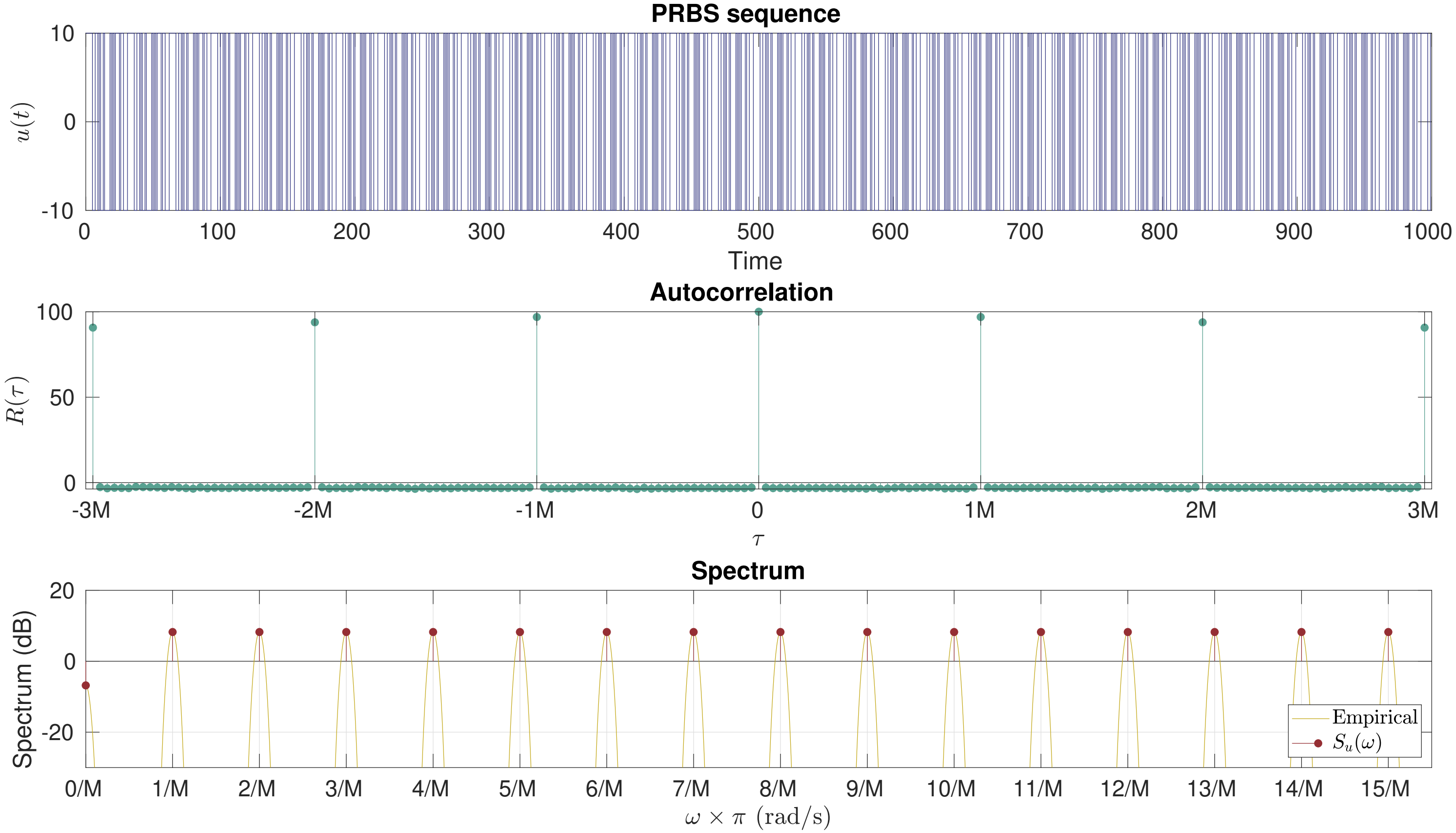
$$C_0 = \frac{a^2}{M^2}, \quad C_k = \frac{a^2}{M^2}(M+1), \quad k = 1, 2, \ldots$$

therefore,

$$S(\omega) = \frac{a^2}{M^2} \left[\delta(\omega) + (M+1) \sum_{k=1}^{M-1} \delta(\omega - 2\pi k/M) \right]$$

it does not resemble spectral characteristic of a white noise (flat spectrum)

autocorrelation and spectrum of PRBS ($n = 5$ and $M = 31$)



$R(\tau) = a^2$ for $\tau = 0, \pm M, \pm 2M, \ldots$ and $R(\tau) = -a^2/M$ otherwise

$$S(\omega) = \frac{a^2}{M^2}\delta(\omega) + \frac{a^2(M+1)}{M^2} \sum_{k=1}^{M-1} \delta(\omega - 2\pi k/M)$$

Comparison of the covariances between filtered inputs

- define $y_1(t)$ as the output of a filter:

$$y_1(t) - ay_1(t-1) = u_1(t),$$

with white noise $u(t)$ of zero mean and variance λ^2

- define $y_2(t)$ be the output of the same filter:

$$y_2(t) - ay_2(t-1) = u_2(t),$$

where $u_2(t)$ is a PRBS of period M and amplitude λ

what can we say about the covariances of $y_1(t)$ and $y_2(t)$?

Comparison of the correlations between filtered inputs

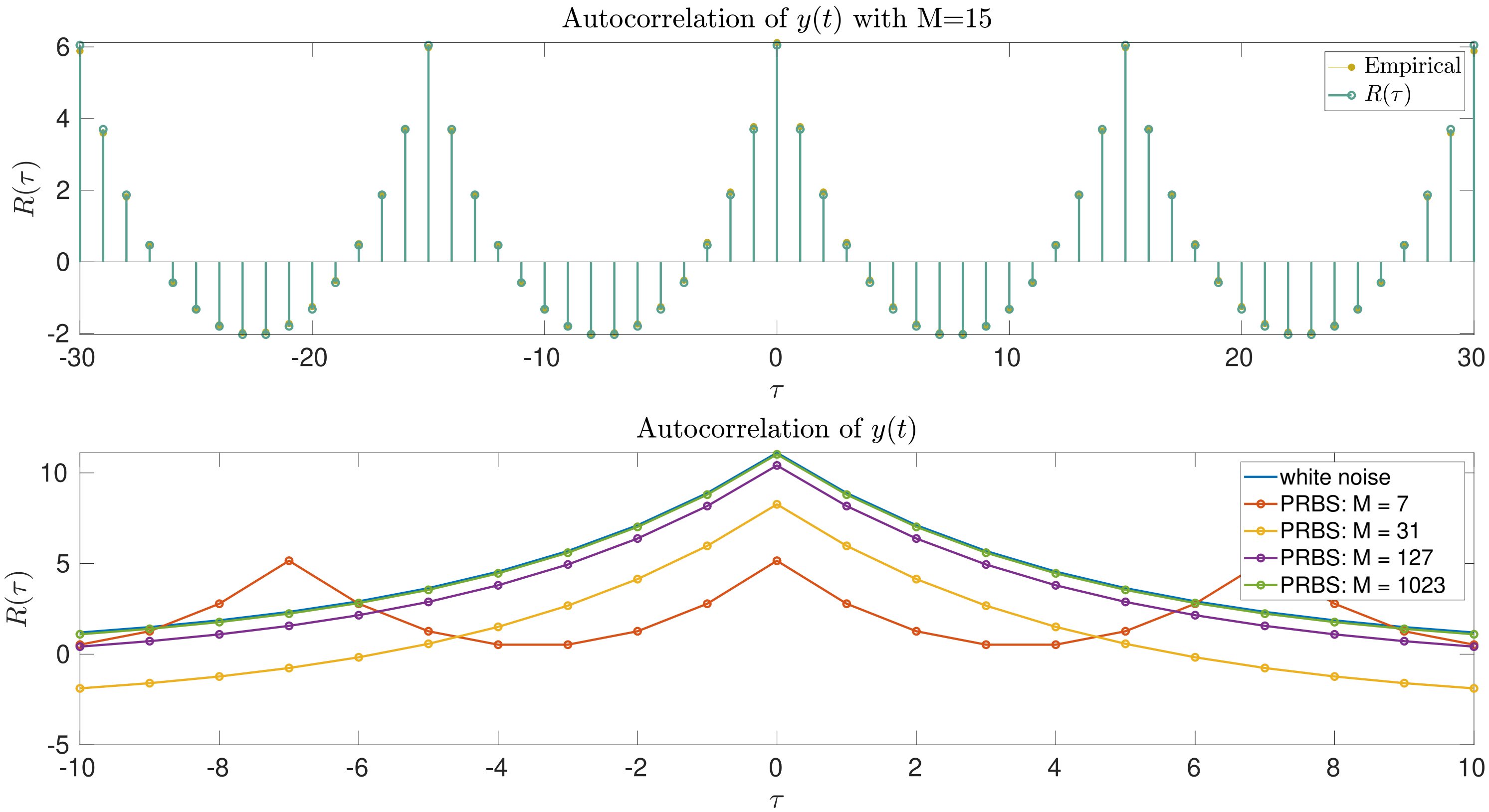
the correlation function of $y_1(t)$ is given by

$$R_1(\tau) = \left(\frac{\lambda^2}{1-a^2}\right) a^{\tau}, \quad \tau \geq 0$$

the correlation function of $y_2(t)$ can be calculated as

$$\begin{aligned} R_2(\tau) &= \int_{-\pi}^{\pi} S_{y_2}(\omega) e^{\mathrm{i}\omega\tau} d\omega \\ &= \int_{-\pi}^{\pi} S_{u_2}(\omega) \left| \frac{1}{1 - ae^{\mathrm{i}\omega}} \right|^2 e^{\mathrm{i}\tau\omega} d\omega \\ &= \frac{\lambda^2}{M} \left[\frac{1}{(1-a)^2} + (M+1) \sum_{k=1}^{M-1} \frac{\cos(2\pi\tau k/M)}{1 + a^2 - 2a\cos(2\pi k/M)} \right] \end{aligned}$$

Plots of the correlation functions



- the filter parameter is $a = 0.8$
- $R(\tau)$ of white noise and PRBS inputs are very close when M is large

Persistent excitation

let $\Delta G = G_2 - G_1$ be the difference between two models with θ_1 and θ_2

$$\Delta \text{MSE} = 0 \quad \Rightarrow \quad |\Delta G(\omega))|^2 S_u(\omega) = 0 \quad \text{almost all frequencies}$$

- when two models yield no difference in MSE, there are two possibilities
  - the two models are not different, or
  - the input spectrum is zero
- we should choose u such that $\Delta\text{MSE} = 0$ implies $\Delta G = 0$
- choose u to be sufficiently *informative* to identify the model

definition: a quasi-stationary process with spectral density $S_u(\omega)$ is said to be **persistently exciting** of order n if

$$|H(\omega)|^2 S_u(\omega) \equiv 0 \quad \Rightarrow \quad H(\omega) \equiv 0 \quad \text{almost all frequencies}$$

for any filter $H(z) = a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}$

Conditions for checking persistent excitation

corollary: if the spectral density matrix of $u(t)$ is *positive definite* at least at n distinct frequencies in $(-\pi, \pi]$ then $u(t)$ has the persistent excitation of order n

lemma: a quasi-stationary signal $u(t)$ is persistently exciting of order n if

1. the limit $R(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u(t+\tau)u(t)^T$ exists

2. the following matrix is positive definite

$$\mathbf{R}(n) = \begin{bmatrix} R(0) & R(1) & \dots & R(n-1) \\ R(-1) & R(0) & \dots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(1-n) & R(2-n) & \dots & R(0) \end{bmatrix}$$

if $u(t)$ is from an ergodic stochastic process, then $\mathbf{R}(n)$ is the usual covariance matrix (assume zero mean)

Examining the order of persistent excitation

- **white noise input** of zero mean and variance λ^2

$$R(\tau) = \lambda^2 \delta(\tau), \quad \Longrightarrow \quad \mathbf{R}(n) = \lambda^2 I_n$$

  thus, white noise is persistently exciting of *all* orders

- **step input** of magnitude λ

$$R(\tau) = \lambda^2, \quad \forall \tau \quad \Longrightarrow \quad \mathbf{R}(n) = \lambda^2 \mathbf{1}_n$$

  a step function is persistently exciting of order 1

- **impulse input**: $u(t) = 1$ for $t = 0$ and 0 otherwise

$$R(\tau) = 0, \quad \forall \tau \quad \Longrightarrow \quad \mathbf{R}(n) = 0$$

  an impulse is *not* persistently exciting of any order

Example : FIR models

a (scalar) FIR model of order $M-1$: $y(t) = \sum_{k=0}^{M-1} h(k)u(t-k)$

can estimated using a correlation analysis

$$R_{yu}(\tau) = \mathbf{E}[y(t+\tau)u(t)] = \sum_{k=0}^{M-1} h(k)R_u(\tau-k)$$

setting $\tau = 0, 1, \ldots, M-1$ gives a set of linear equations in $h(k)$

$$\begin{bmatrix} R_{yu}(0) \\ R_{yu}(1) \\ \vdots \\ R_{yu}(M-1) \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(1) & \cdots & R_u(M-1) \\ R_u(-1) & R_u(0) & \cdots & R_u(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_u(1-M) & R_u(2-M) & \cdots & R_u(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \end{bmatrix}$$

- the equations has a unique solution iff $\mathbf{R}(M)$ is nonsingular (u is p.e. of order M)
- need more p.e. order if the model is more complex

Properties of persistently exciting signals

assumptions:

- $u(t)$ is a multivariable ergodic process
- $S_u(\omega)$ is positive in at least n distinct frequencies within $(-\pi, \pi)$

we have the following two properties

Property 1 $u(t)$ is persistently exciting of order n

Property 2 if $H(z)$ is an asymptotically stable linear filter and $\det H(z)$ has no zero on the unit circle then the filtered signal $y(t) = H(q^{-1})u(t)$ is persistently exciting of order n

we can imply an ARMA process is persistently exciting of *any finite order*

Examining the order of PRBS

consider a PRBS of period M and magnitude $a, -a$

the matrix containing n-covariance sequences (where $n \leq M$) is

$$\mathbf{R}(n) = \begin{bmatrix} a^2 & -a^2/M & \ldots & -a^2/M \\ -a^2/M & a^2 & \ldots & -a^2/M \\ \vdots & \vdots & \ddots & \vdots \\ -a^2/M & -a^2/M & \ldots & a^2 \end{bmatrix}$$

for any $x \in \mathbf{R}^n$,

$$\begin{aligned} x^T\mathbf{R}(n)x &= x^T\left[\left(a^2 + \frac{a^2}{M}\right)I - \frac{a^2}{M}\mathbf{1}\mathbf{1}^T\right]x \\ &\geq a^2\left(1 + \frac{1}{M}\right)x^Tx - \frac{a^2}{M}x^Tx\mathbf{1}^T\mathbf{1} = a^2\|x\|^2\left(1 + \frac{(1-n)}{M}\right) \geq 0 \end{aligned}$$

a PRBS with period M is persistently exciting of order M

Examining the order of sum of sinusoids

consider the signal $u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$

where $0 \leq \omega_1 < \omega_2 < \ldots < \omega_m \leq \pi$

the spectral density of u is given by

$$S(\omega) = \sum_{k=1}^{m} \frac{C_k}{2}[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)]$$

therefore $S(\omega)$ is nonzero (in the interval $(-\pi, \pi]$) in exactly n points where

$$n = \begin{cases} 2m, & 0 < \omega_1, \omega_m < \pi \\ 2m-1, & 0 = \omega_1, \text{ or } \omega_m = \pi \\ 2m-2, & 0 = \omega_1 \text{ and } \omega_m = \pi \end{cases}$$

it follows from Property 1 that $u(t)$ is persistently exciting of order n

Summary

- the choice of input is imposed by the type of identification method
- the input signal should be persistently exciting of a certain order to ensure that the system of a certain order can be identified
- some often used signals include PRBS and ARMA processes

References

Chapter 5 in

T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989

Chapter 4 in

L. Ljung, *System Identification: Theory for the User*, 2nd edition, Prentice Hall, 1999