3. Canonical forms

- coordinate transformation
- controllable canonical form
- observable canonical form
- controller canonical form
- observer canonical form

Coordinate transformation

let the state-space equation of a system be

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

if we transform the state x into a new state z by

$$z = T^{-1}x \qquad \text{or} \qquad x = Tz$$

the new state-space equation is

$$\dot{z} = \overline{A}z + \overline{B}u, \quad y = \overline{C}z + \overline{D}u$$

$$\overline{A} = T^{-1}AT, \quad \overline{B} = T^{-1}B, \quad \overline{C} = CT, \quad \overline{D} = D$$

controllability and observability matrices:

$$\mathcal{C} = T\overline{\mathcal{C}}, \qquad \mathcal{O}T = \overline{\mathcal{O}}$$

Controllable canonical form

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}$$

- the controllability matrix is $\mathcal{C} = I_n$
- (A, B) is controllable

assumption: the system (A, B) is controllable

goal: find a transformation T such that $z = T^{-1}x$ is in controllable form:

$$\overline{A} = T^{-1}AT, \qquad \text{and} \qquad \overline{B} = T^{-1}B$$

have controllable companion form as in page 3-3

solution: from $\mathcal{C} = T\overline{\mathcal{C}}$, since $\overline{\mathcal{C}} = I$ we must have $T = \mathcal{C}$

another way to prove this is to assume T takes the form

$$T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$$

- from $B = T\overline{B}$, we have $B = t_1$

- from $AT = T\overline{A}$,

$$\begin{bmatrix} At_1 & At_2 & \cdots At_n \end{bmatrix} = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & & 0 & -a_{n-2} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -a_1 \end{bmatrix}$$

$$= \begin{bmatrix} t_2 & t_3 & \cdots & -t_1 a_n - t_2 a_{n-1} \cdots - t_n a_1 \end{bmatrix}$$

from which we have

$$t_1 = B, \quad t_2 = At_1 = AB, \quad t_3 = At_2 = A^2B \quad \cdots \quad t_n = A^{n-1}B$$

conclusion: T that transforms a system into the controllable form is

$$T = \mathcal{C}$$

(the controllability matrix of the original system)

Uncontrollable systems

consider an n-dimensional uncontrollable system of the form

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix} z$$

- assume $\mathbf{rank}(\overline{\mathcal{C}}) = r < n$
- the controllability matrix has the form

$$\overline{\mathcal{C}} = \begin{bmatrix} \overline{B}_1 & \overline{A}_{11}\overline{B}_1 & \cdots & \overline{A}_{11}^{r-1}\overline{B}_1 & \cdots & \overline{A}_{11}^{n-1}\overline{B}_1 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

- $z_1 \in \mathbf{R}^r$ is controllable but $z_2 \in \mathbf{R}^{n-r}$ is not
- the transfer function is $\overline{H}(s) = \overline{C}_1(sI - \overline{A}_{11})^{-1}\overline{B}_1$

how can we transform an uncontrollable system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

using $z = T^{-1}x$ where z has the form in page 3-6 ?

from $\mathcal{C} = T\overline{\mathcal{C}}$ and assume $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$

$$T\overline{\mathcal{C}} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \mathcal{C}(\overline{A}_{11}, \overline{B}_1) & \text{XX} \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{cccc|c} B & AB & \cdots & A^{r-1}B & \text{XX} \end{array} \right]$$

where XX is some nonzero term that is not relevant to our calculation

- $\mathcal{C}(\overline{A}_{11}, \overline{B}_1) = I$ if $(\overline{A}_{11}, \overline{B}_1)$ is in controllable form
- hence, $T_1 = \begin{bmatrix} B & AB & \cdots A^{r-1}B \end{bmatrix}$ (the first r columns of $\mathcal{C}$)
- then choose the columns of T_2 to be any vectors that are independent of the columns in T_1

Observable canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- the observability matrix is $\mathcal{O} = I_n$
- (A, C) is observable

assumption: the system (A, C) is observable

goal: find a transformation T such that $z = T^{-1}x$ is in observable form:

$$\overline{A} = T^{-1}AT, \qquad \text{and} \qquad \overline{B} = T^{-1}B$$

have observable companion form as in page 3-8

solution: from $\mathcal{O}T = \overline{\mathcal{O}}$, since $\overline{\mathcal{O}} = I$ we must have

$$T = \mathcal{O}^{-1}$$

T is the inverse of the observability matrix of the original system

Unobservable systems

consider an n-dimensional unobservable system of the form

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} u, \quad y = \begin{bmatrix} \overline{C}_1 & 0 \end{bmatrix} z$$

- assume $\mathbf{rank}(\overline{\mathcal{O}}) = r < n$
- the observability matrix has the form

$$\overline{\mathcal{O}} = \begin{bmatrix} \overline{C}_1 & 0 \\ \vdots & \vdots \\ \overline{C}_1 \overline{A}_{11}^{r-1} & 0 \\ \vdots & \vdots \\ \overline{C}_1 \overline{A}_{11}^{n-1} & 0 \end{bmatrix} \quad \triangleq \quad \left[\begin{array}{c|c} \mathcal{O}(\overline{A}_{11}, \overline{C}_1) & 0 \\ \hline \mathsf{XX} & 0 \end{array}\right]$$

- $z_1 \in \mathbf{R}^r$ is observable but $z_2 \in \mathbf{R}^{n-r}$ is not
- the transfer function is $\overline{H}(s) = \overline{C}_1 (sI - \overline{A}_{11})^{-1} \overline{B}_1$

how can we transform an unobservable system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

using $z = T^{-1}x$ where z has the form in page 3-10 ?

from $\mathcal{O}T = \overline{\mathcal{O}}$, if we define

$$W = T^{-1} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

then we can write $\mathcal{O} = \overline{\mathcal{O}}W$ as

$$\left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{r-1} \\ \hline \text{XX} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{O}(\overline{A}_{11}, \overline{C}_1) & 0 \\ \hline \text{XX} & 0 \end{array}\right] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

hence,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} = \mathcal{O}(\overline{A}_{11}, \overline{C}_1) W_1$$

- $\mathcal{O}(\overline{A}_{11}, \overline{C}_1) = I$ if $(\overline{A}_{11}, \overline{C}_1)$ is in observable form
- hence, $W_1 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}$ (the first r rows of $\mathcal{O}$)
- then choose the rows of W_2 to be any vectors that are independent of the rows in W_1

Controller canonical form

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}$$

- $\mathcal{C}$ is an upper triangular matrix with 1's on the diagonal
- (A, B) is controllable
- $A - BK$ preserves the canonical structure with any feedback gain

$$K = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$$

Observer canonical form

$$
\begin{aligned}
A &= \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -a_{n-1} & 0 & 0 & & 1 & 0 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}
\end{aligned}
$$

- $\mathcal{O}$ is a lower triangular with 1's on the diagonal
- (A, C) is observable
- $A - LC$ preserves the canonical structure with any observer gain

$$L = \begin{bmatrix} l_1 & l_2 & \dots & l_n \end{bmatrix}^T$$

References

Chapter 2 in

T. Kailath, *Linear Systems*, Prentice-Hall, 1980

Chapter 5 in

D. Banjerdpongchai, *Dynamical Control Systems: Analysis, Design and Applications*, Chulalongkorn University Press, 2008