8. LMI in Control Theory

- the positive-real Lemma
- the bounded-real Lemma
- stabilization
- constraints on input
- decay rate
- reachable set with unit-energy inputs
- bound on output energy

Positive-real lemma

a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is said to be **passive** if

$$\int_0^T u(t)^T y(t) dt \geq 0$$

for all solutions $x(t)$ with $x(0) = 0$.

passivity is equivalent to the transfer matrix H being **positive-real**:

$$H(s) + H(s)^* \geq 0, \quad \text{for all } \operatorname{Re}(s) > 0$$

where

$$H(s) = C(sI - A)^{-1}B + D$$

assume A is stable and (A, B, C) is minimal

the linear system is *passive* if and only if the LMI

$$P \succ 0, \quad \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \preceq 0$$

is feasible

if we assume $D + D^T \succ 0$, the above LMI is equivalent to

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T \preceq 0$$

the LMI is feasible if and only if there exists $P \succ 0$ such that

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T = 0$$

(just the equality substituted)

Bounded-real Lemma

a linear system is said to be **nonexpansive** if

$$\int_0^T y(t)^T y(t)dt \leq \int_0^T u(t)^T u(t)dt$$

for all solutions $x(t)$ with $x(0) = 0$

nonexpansitivity is equivalent to the transfer matrix H is *bounded*:

$$H(s)^* H(s) \leq I, \quad \text{for all } \operatorname{Re}(s) > 0$$

this is sometimes expressed as $\|H\|_\infty \leq 1$ where

$$\|H\|_\infty \triangleq \sup\{\|H(s)\| \mid \operatorname{Re}(s) > 0\}$$

is called the $\mathbf{H}_\infty$ norm of the transfer matrix H

assume A is stable and (A, B, C) is minimal

a linear system is nonexpansive if and only if the LMI

$$P \succ 0, \quad \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I \end{bmatrix} \preceq 0$$

is feasible

furthermore, if $D^T D \prec I$

the LMI is feasible if and only if the ARE

$$A^T P + PA + C^T C + (PB + C^T D)(I - D^T D)^{-1}(PB + C^T D)^T = 0$$

has a real solution $P = P^T$

(again, substitute the inequality with equality)

Stabilizability

the pair (A, B) is *stabilizable* if there exists K such that

$$A + BK \text{ is stable}$$

the closed-loop system is stable if and only if there exists $P \succ 0$

$$(A + BK)^T P + P(A + BK) \prec 0$$

or equivalently, there exists $Q \succ 0$ such that

$$Q(A + BK)^T + (A + BK)Q \prec 0$$

this is not LMI in Q, K but by a change of variable $Y = KQ$, we have

$$AQ + QA^T + BY + Y^T B^T \prec 0$$

which is an LMI in Q and Y

Holdable ellipsoids

we say that the ellipsoid

$$\mathcal{E} = \{x \in \mathbf{R}^n \mid x^T Q^{-1} x \leq 1\}$$

is *holdable* for the system $\dot{x} = Ax + Bu$ if there exists K such that

$\mathcal{E}$ is *invariant* for the closed-loop system with $u = Kx$

example: existence of a quadratic Lyapunov function that is decreasing:

$$V(t) = x(t)^T Q^{-1} x(t) \leq x(0)^T Q^{-1} x(0), \quad \text{for all } t$$

therefore Q from the LMI in page 8-6 characterizes holdable ellipsoids

Constraints on the control input

design $u = Kx$ by picking Q, Y that satisfy the stabilizability condition

in addtion, if $x(0)^T Q^{-1} x(0) \leq q$ then $x(t)$ belongs to the ellipsoid

$$\mathcal{E} = \left\{ x \mid x^T Q^{-1} x \leq 1 \right\}$$

for all t

we will derive LMIs that guarantee a constraint on input norm

- 2-norm

$$\max_{t \geq 0} \|u(t)\|_2 \leq \mu$$

- ∞-norm

$$\max_{t \geq 0} \|u(t)\|_\infty \leq \mu$$

2-norm constraint on the control input

we can derive the maximum 2-norm of u by

$$\begin{aligned}\max_{t\geq 0}\|u(t)\|_2 &= \max_{t\geq 0}\|YQ^{-1}x(t)\|_2 \leq \max_{x\in\mathcal{E}}\|YQ^{-1}x\|_2 \\ &= \sqrt{\lambda_{\max}(Q^{-1/2}Y^TYQ^{-1/2})}\end{aligned}$$

note that if $A \preceq \alpha I$ then $\lambda_{\max}(A) \leq \alpha$

therefore the constraint $\|u\|_2 \leq \mu$ is enforced for all t if

$$x(0)^TQ^{-1}x(0) \leq 1, \quad Q^{-1/2}Y^TYQ^{-1/2} \preceq \mu^2 I$$

which can be expressed in LMI as

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & Q \end{bmatrix} \succeq 0, \quad \begin{bmatrix} Q & Y^T \\ Y & \mu^2 I \end{bmatrix} \succeq 0$$

hold, where Q, Y satisfies the stabilizability conditions

Constraint on magnitude peak of the input

we can derive a bound on ∞-norm of u as

$$\begin{aligned}
\max_{t\geq 0} \|u(t)\|_\infty &= \max_{t\geq 0} \|YQ^{-1}x(t)\|_\infty \\
&\leq \max_{x\in\mathcal{E}} \|YQ^{-1}x\|_\infty \\
&= \max_i \sqrt{(YQ^{-1}Y^T)_{ii}}
\end{aligned}$$

Recall: the last step follows from the fact that

- let $a_1, \ldots, a_n^T$ be row vectors of a matrix A
- $\|Ax\|_\infty$ is maximized over $\|x\|_2 \leq 1$ when $x = a_k/\|a_k\|$
- k is the index such that $\|a_k\|_2$ is maximum
- hence $\max_{\|x\|_2\leq 1} \|Ax\|_\infty = \max_k \|a_k\|_2 = \max_k \sqrt{(AA^T)_{kk}}$

therefore, the constraint $\|u(t)\|_\infty \leq \mu$ for $t \geq 0$ if

$$x(0)^T Q^{-1} x(0), \quad (YQ^{-1}Y^T)_{ii} \leq \mu^2, \quad \text{for all } i$$

which can be represented in LMI as

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & Q \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \succeq 0, \quad X_{ii} \leq \mu^2$$

where $Q \succ 0$ and Y satisfy the stabilization conditions

Decay rate

define $V = z^T P z$ and if

$$\dot{V}(x) \leq -2\alpha V(x)$$

for all trajectories, then

$$V(x(t)) \leq V(x(0))e^{-2\alpha t} \quad \Longrightarrow \quad \|x(t)\| \leq e^{-\alpha t}\sqrt{\kappa(P)}\|x(0)\|$$

therefore, the decay rate is at least α

the condition $\dot{V}(x) \leq -2\alpha V(x)$ is equivalent to

$$A^T P + PA + 2\alpha P \preceq 0$$

the largest decay rate can be found by solving

$$\begin{array}{ll} \text{maximize} & \alpha \\ \text{subject to} & P \succ 0 \\ & A^T P + PA + 2\alpha P \preceq 0 \end{array}$$

Reachable sets with unit-energy inputs

consider a linear system

$$\dot{x} = Ax + Bu$$

and all trajectories that start from $x(0) = 0$

denote $\mathcal{R}_u$ the set of reachable states with unit-energy input

$$\mathcal{R}_u \triangleq \left\{ x(T) \;\middle|\; \int_0^T u(t)^T u(t) dt \leq 1, \;\; T \geq 0 \right\}$$

we can bound $\mathcal{R}_u$ by ellipsoids of the form

$$\mathcal{E} = \left\{ x \mid x^T P x \leq 1 \right\}, \quad P \succ 0$$

we can show that $\mathcal{E}$ contains the reachable set $\mathcal{R}_u$

suppose $V(z) = z^T P z$ satisfies

$$\dot{V} \leq u^T u, \quad \text{for all } x, u$$

integrating both sides gives

$$V(x(T)) - V(x(0)) \leq \int_0^T u^T u \, dt$$

since $V(x(0)) = 0$, we have

$$x(T)^T P x(T) = V(x(T)) \;\leq\; \int_0^T u^T u \, dt \;\leq\; 1$$

for every $T \geq 0$ and every input u such that $\int_0^T u^T u \, dt \leq 1$

the condition $\dot{V} \leq u^T u$ is equivalent to

$$P \succ 0, \quad \begin{bmatrix} A^T P + PA & PB \\ B^T P & -I \end{bmatrix} \preceq 0$$

which can be expressed as an LMI in $Q = P^{-1}$

$$Q \succ 0, \quad AQ + QA^T + BB^T \preceq 0$$

for LTI systems, the ellipsoid bound is sharp with $P = W_c^{-1}$ where

$$W_c \triangleq \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

is the controllability Grammian and W_c satisfies

$$AW_c + W_c A^T + BB^T = 0$$

Bounds on output energy

we seek the maximum output energy given a certain initial state

$$\max \left\{ \int_0^\infty y^T y dt \,\middle|\, \dot{x} = Ax, \quad y = Cx \right\}$$

where $x(0)$ is given and the maximum is taken over A, C

suppose there exists $V = z^T P z$ such that

$$P \succ 0, \quad \dot{V}(x) \leq -y^T y, \quad \text{for every } x, y$$

integrating both sides gives

$$V(x(T)) - V(x(0)) \leq -\int_0^T y^T y dt$$

since $V(x(T)) \geq 0$, then $V(x(0))$ is an upper bound on the maximum energy of y

the condition

$$P \succ 0, \quad \dot{V}(x) \leq -y^T y, \quad \text{for every } x, y$$

is equivalent to

$$P \succ 0, \quad A^T P + PA + C^T C \preceq 0$$

therefore, we can obtain the best upper bound by solving

$$\begin{array}{ll} \text{minimize} & x(0)^T P x(0) \\ \text{subject to} & P \succ 0, \qquad A^T P + PA + C^T C \preceq 0 \end{array}$$

again the bound is sharp for LTI systems and the solution is given by

$$P = W_o, \quad W_o \triangleq \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

and W_o satisfies the Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0$$

References

S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, 1994.