7. Linear Quadratic Gaussian Control

- output feedback
- Kalman filter
- LQG/LQR

Output feedback

consider a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

a state-feedback controller has a form

$$u(t) = -Kx(t)$$

which requires the availability of the process measurement

when the state variables are not accessible, one can use

$$u(t) = -K\hat{x}(t)$$

where $\hat{x}(t)$ is an estimate of $x(t)$ based on the output y

Full-order observers

x cannot be fully measured and the goal is to estimate x based on y

our approach is to replicate the process dynamic in $\hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu$$

define the state estimation error $e = x - \hat{x}$, we can see

$$\dot{e} = Ax - A\hat{x} = Ae$$

- if A is stable, then the error goes to zero asymptotically
- if A is unstable, e is unbounded and $\hat{x}$ grows further apart from x

to avoid this problem, one can consider a correction term as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{x}(0) = 0$$

(feed y back to the estimator)

L is a given matrix, called *observer gain* matrix

the error dynamic now is

$$\dot{e} = Ax - A\hat{x} - L(Cx - C\hat{x}) = (A - LC)e, \quad e(0) = x(0)$$

- the observer error goes to zero if L is chosen such that

$$A - LC \text{ is stable}$$

- we can make e goes to zero fast if the eigenvalues of $A - LC$ can be arbitrarily assigned
- eigenvalues of $A - LC$ are same as those of $(A^T - C^T L^T)$
- hence, choosing L is the dual of state-feedback design problem for the pair (A^T, C^T)
- eigenvalues of $A - LC$ can be freely reassigned if and only if (A, C) observable

Observer-based controller

closed-loop equations after a state feedback

$$u(t) = r(t) - K\hat{x}(t)$$

where $r(t)$ is reference input, is

$$\begin{aligned} \dot{x} &= Ax - BK\hat{x} + Br \\ \dot{\hat{x}} &= A\hat{x} + L(y - \hat{y}) + Br - BK\hat{x} \end{aligned}$$

or

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r$$

Separation principle

change the coordinate

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix}$$

the dynamics in the new coordinate is

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} r$$

- the dynamic of e does not depend on x
- closed-loop eigenvalues are

$$\{\mathrm{eig}(A - BK)\} \cup \{\mathrm{eig}(A - LC)\}$$

- one can assign eigenvalues of the system and the observer *independently*

Observer of noisy system

in general, the system is corrupted by noise

$$\dot{x} = Ax + Bu + w, \quad y = Cx + v$$

where w is process noise and v is measurement noise

rewrite the error dynamic of an observer

$$\begin{aligned} \dot{e} &= Ax + Bu + w - A\hat{x} - Bu - L(Cx + v - C\hat{x}) \\ &= (A - LC)e + w - Lv \end{aligned}$$

- due to w, v, the estimation will generally not go to zero
- one would like the error to remain small by a good choice of L
- the optimal choise of L is given by the **Kalman gain**

Kalman filter

assume w, v are uncorrelated zero-mean Gaussian white noise, $i.e.$,

$$\mathbf{E}\, w(t)w(\tau)^* = W\delta(t-\tau), \quad \mathbf{E}\, v(t)v(\tau)^* = V\delta(t-\tau)$$
$$\mathbf{E}\, w(t)v(\tau)^* = 0$$

spectral density matrices of w, v are

$$W \succeq 0, \quad \text{and} \quad V \succ 0, \quad \text{respectively}$$

The optimal observer gain which minimizes $\mathbf{E}\|e(t)\|^2$ is

$$L = PC^*V^{-1}$$

where P is the unique positive-semidefinite solution of the ARE

$$PA^* + AP - PC^*V^{-1}CP + W = 0$$

when using the optimal gain, this system is called the **Kalman-Bucy** filter

$A - LC$ is stable as long as the two conditions hold

- (A, C) is observable
- (A, W) is controllable

Recall: LQR control with $J = \int_0^\infty x(t)^* Q x(t) + u^*(t) R u(t) dt$

we need to solve ARE

$$PA + A^*P - PBR^{-1}B^*P + Q = 0$$

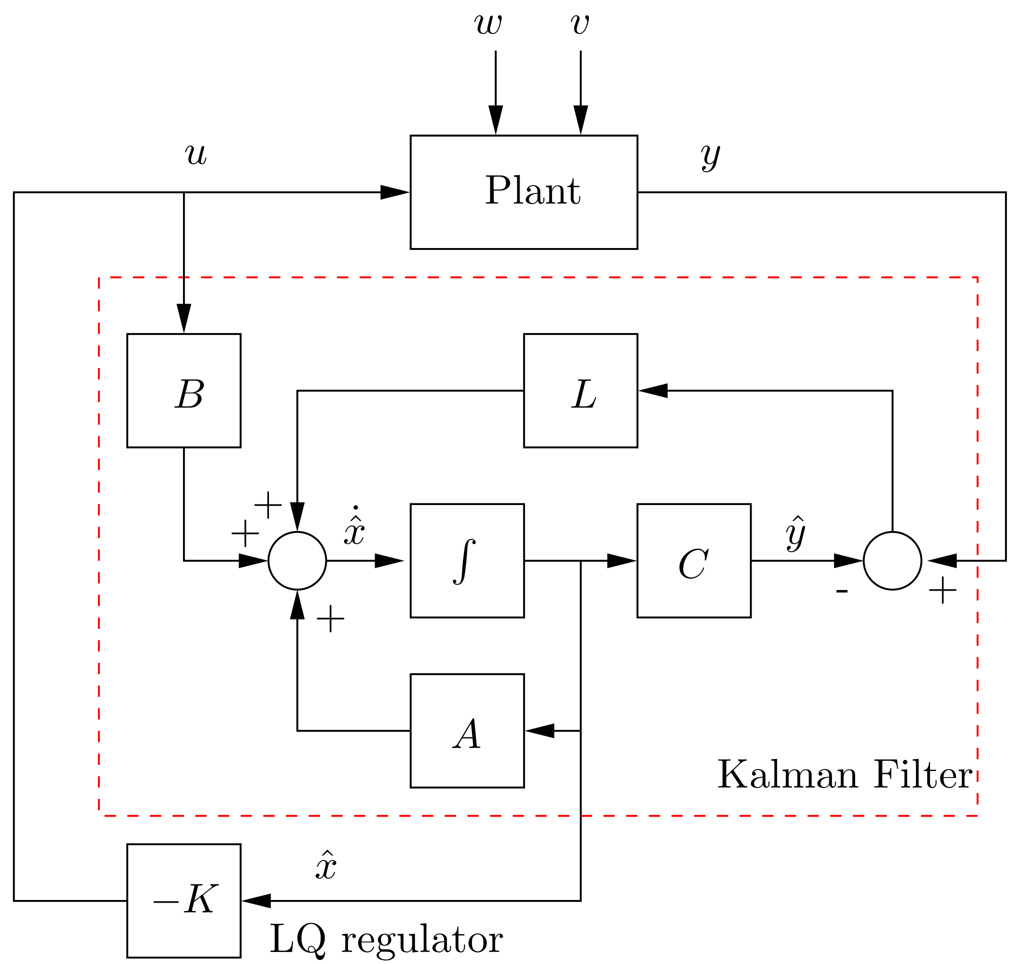
with conditions

- (A, B) controllable to guarantee $J_{\min} < \infty$
- (A, Q) observable to obtain a unique positive definition P

Duality: Kalmain filter is equivalent to designing an LQR controller on the dual system (A^*, C^*, B^*, D^*) with $Q = W$, $R = V$

Linear Quadratic Gaussian Control

a combination of optimal state estimation and optimal state feedback



Example

we design an LQG controller to a spring-mass system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + v$$

where covariances of w are v are I and 0.1 respectively

we use the parameters

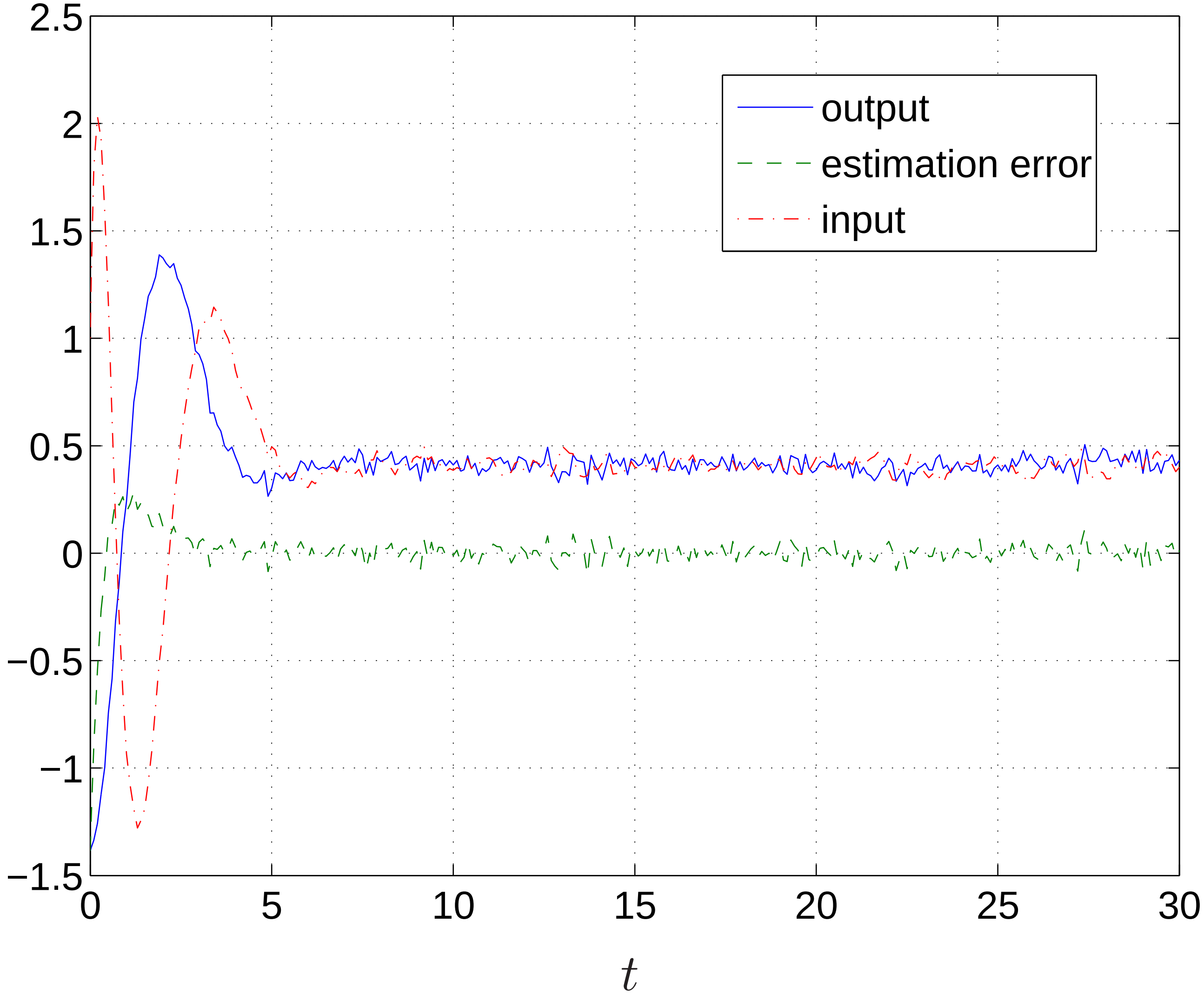
$$Q = 5C^*C, \quad R = 1, \quad W = I, \quad V = 0.1$$

(A, C) observable and (A, W) controllable

MATLAB codes

```
n = 2; A = [0 1;-1 0]; B = [0 1]'; C = [1 0];
Q = 5*C'*C; R = 1;
K = lqr(A,B,Q,R);
W = eye(n); V = 0.1;
L = lqe(A,eye(n),C,W,V);
```

closed-loop response to unit step in reference



(a standard LQG does not have an integral action)

Robustness properties

LQR-controlled systems if R is diagonal, then

the system will have a gain margin equal to ∞ and a phase margin of 60°

The input $u = -Kx$ can have a complex perturbation

$$\Delta u_i = k_i e^{j\theta_i}, \quad i = 1, 2, \ldots, m \quad m = \text{number of inputs}$$

without causing instability providing

1. $\theta_i = 0$ and $0.5 \leq k_i \leq \infty, \quad i = 1, 2, \ldots, m$
2. $k_i = 1$ and $|\theta_i| \leq 60^\circ, \quad i = 1, 2, \ldots, m$

LQG-controlled systems LQG-controlled system with a combined Kalman filter and LQR control law, there are NO guaranteed stability margins

Counterexample

we will show that the gain margin of LQG becomes very small as feedback and observer gains become large

consider an unstable SISO system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + v \end{aligned}$$

with parameters in controller and observer design given by

$$Q = \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad R = 1, \quad W = \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad V = 1$$

ARE for controller design has the positive definite solution

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad p_{12} = p_{22} = 2 + \sqrt{4+\alpha}, \quad p_{11} = (p_{12}^2 - \alpha)/2$$

the LQR gain is

$$K = -R^{-1}B^*P = (2 + \sqrt{4+\alpha}) \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \triangleq \quad \begin{bmatrix} k & k \end{bmatrix}$$

solving the ARE for observer design, the Kalman gain is

$$L = (2 + \sqrt{2+\beta}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} l \\ l \end{bmatrix}$$

the equations of the controlled system and the observer are

$$\begin{aligned} \dot{x} &= Ax - \tilde{B}K\hat{x} + Fw \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(Cx + v - C\hat{x}) \end{aligned}$$

where

- $\tilde{B}$ is the actual input matrix for the system ($\tilde{B}$ contains an uncertainty)
- B is the input matrix used in control design

we're concerned if the LQG has a robust stability (from uncertainty in $\tilde{B}$)

assume $\tilde{B} = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$ where $\mu \neq 1$ (so $\tilde{B} \neq B$)

the eigenvalues of the dynamic matrix

$$\begin{bmatrix} A & \tilde{B}K \\ LC & (A - LC + BK) \end{bmatrix}$$

satisfies the characteristic equations of the form

$$s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0 = 0$$

and from Routh's criterion, all c_k's must be positive however, we can show that

$$c_1 = l + k - 4 + 2(\mu - 1)kl, \quad c_0 = 1 + (1 - \mu)kl$$

and the value of μ can make these two coefficients to change sign:

- if $\mu > (1 + 1/kl)$ then $c_0 < 0$
- if $\mu < [1 - (l + k - 4)/2lk]$ then $c_1 < 0$

so the gain margin is smaller as kl becomes larger

- increasing l and k relates to increasing Q and W
- robustness is enhanced by low weighting on the state and covariance of process noise

References

Lecture note on

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S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, John Wiley & Sons, 1996

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