5. Observer-based Controller Design

- state feedback
- pole-placement design
- regulation and tracking
- state observer
- feedback observer design
- LQR and LQG

State feedback

consider an LTI system

continuous-time

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

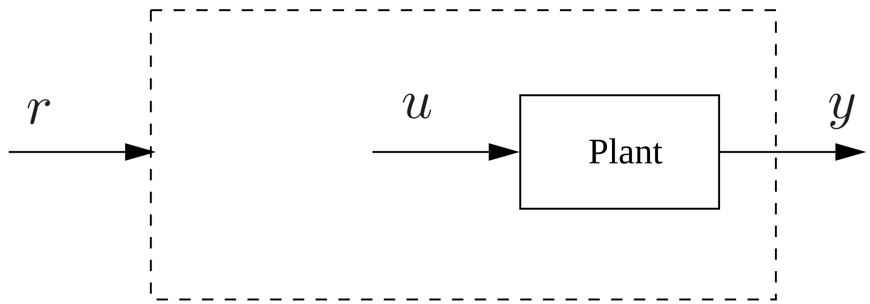
discrete-time

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

Problem: design u such that the system behaves as desired, for example:

- stabilize the system (if A is unstable)
- make the output track a reference faster
- make the output small while use less energy of u

- r: reference signal
- u: control/actuating signal
- y: plant output/controlled signal



types of control

- open-loop control: u depends only on r and is independent of y
- closed-loop control: u depends on both r and y

state feedback u is a function of state variables, *i.e.*, $u = f(x, t)$ for some function f

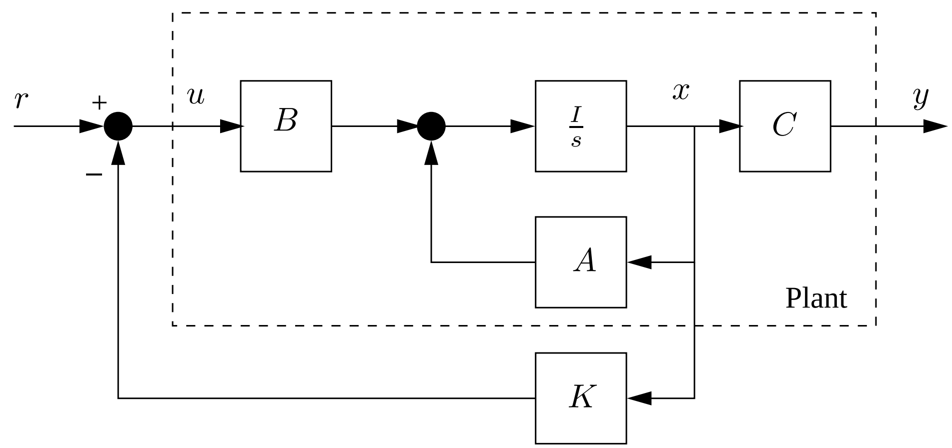
Linear state feedback

we study a **constant linear** state feedback:

$$u(t) = r(t) - Kx(t)$$

r is a reference input and K is called a **feedback gain**

then the closed-loop system is $\dot{x} = (A - BK)x + Br$



eigenvalues of $(A - BK)$ determine the behavior of the closed-loop system

Controllability under a state feedback

Fact: $(A - BK, B)$ is controllable if and only if (A, B) is controllability

Proof: suppose (A, B) is not controllable, $i.e.$, $\exists w \neq 0$ such that

$$w^* A = \lambda w^*, \; w^* B = 0 \quad \Longleftrightarrow \quad w^*(A - BK) = \lambda w^*, \; w^* B = 0$$

hence, $(A - BK, B)$ is also not controllable

- the controllability property is *invariant* under any state feedback
- eigenvalues of $(A - BK)$ can be arbitrarily assigned provided that complex conjugate eigenvalues are assigned in pairs
- what about the observability property ?

Coordinate transformation

consider a linear transformation $z = T^{-1}x$ where

$$\dot{x} = (A - BK)x + Br$$

the dynamics in the new coordinate is

$$\dot{z} = (\bar{A} - \bar{B}\bar{K})z + \bar{B}r$$

where

$$\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B, \quad \bar{K} = KT$$

Pole-placement design

Fact: $\lambda(A - BK)$ cannot be freely reassigned if (A, B) is **not** controllable

if (A, B) is not controllable then we can put it in uncontrollable form

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

consider a feedback gain $\overline{K}$ and partition it as

$$\bar{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

then the closed-loop dynamic matrix is

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} A_{11} - B_1 K_1 & A_{12} - B_1 K_2 \\ 0 & A_{22} \end{bmatrix}$$

$\lambda(A_{22})$ cannot be moved, so uncontrollable modes remain uncontrollable

Pole placement for single-input systems

Fact: $\lambda(A - BK)$ can be freely reassigned if (A, B) is controllable

- change coordinate to the **controller canonical form**

$$\bar{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- assume $\bar{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$, so

$$\det(sI - \bar{A} + \bar{B}\bar{K}) = s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_n + k_n)$$

- choose the closed-loop poles arbitrarily by a suitable choice of $\bar{K}$
- transform $\bar{K}$ back to the new original coordinate

comments:

- drastic change in a desired characteristic polynomial requires a large K
- zeros of $C(sI-A)^{-1}B$ are the same as that of $C(sI-A+BK)^{-1}B$

  suppose (A,B,C) is in the controller form

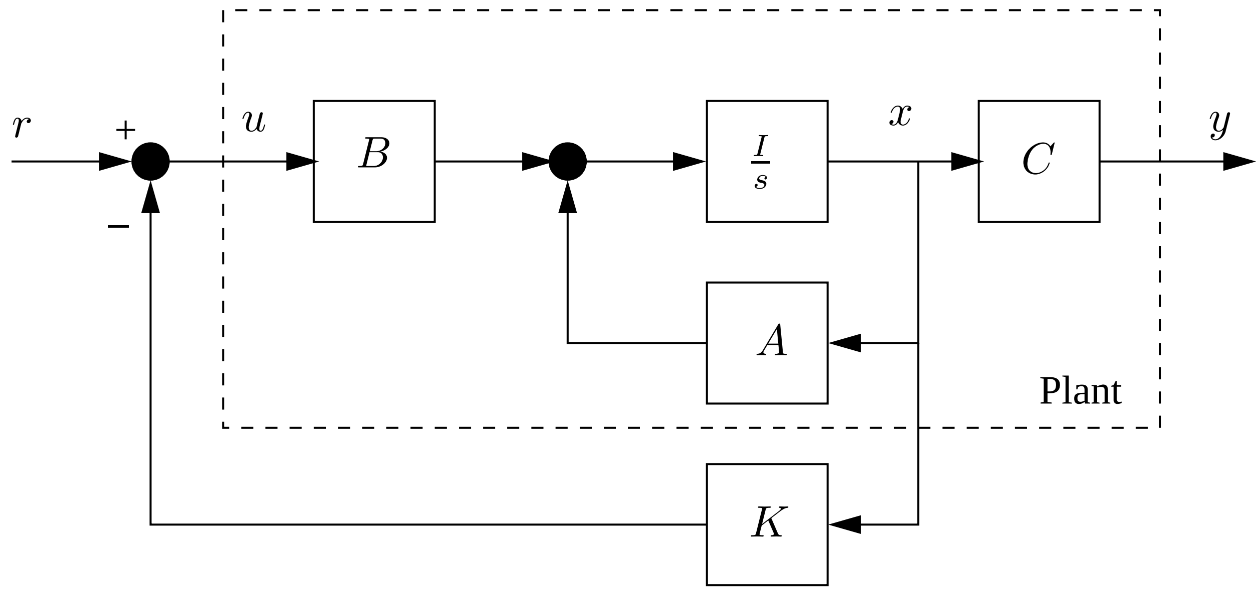
$$
\begin{aligned}
C(sI-A)^{-1}B &= \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \\
C(sI-A+BK)^{-1}B &= \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n}
\end{aligned}
$$

- the zeros of the transfer function from r to y are not affected by K
- state feedback can result in unobservable modes due to cancellation in

$$
\frac{C\,\mathrm{Adj}(sI-A+BK)B}{\det(sI-A+BK)}
$$

Regulator problem

consider the state feedback configuration with $r = 0$

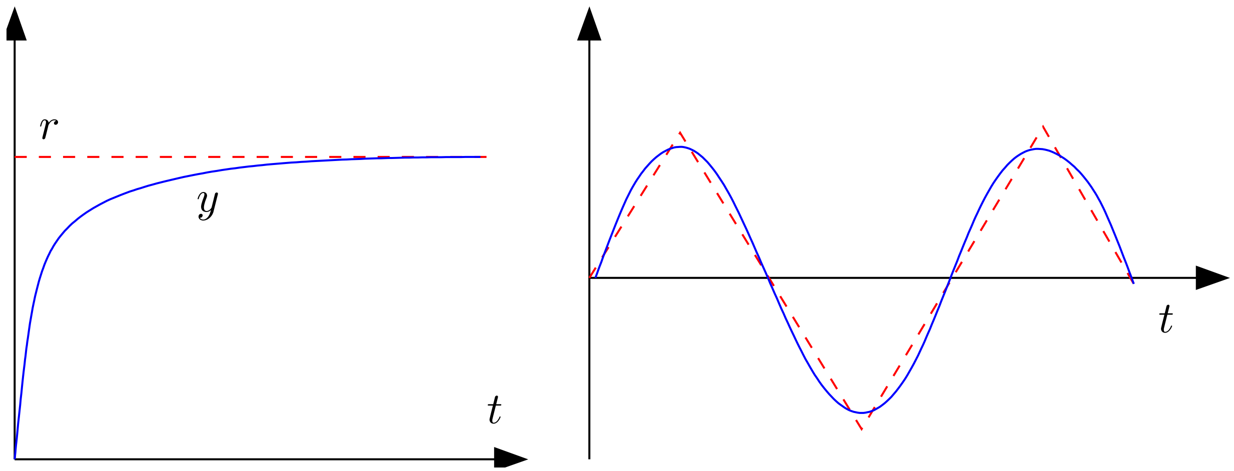


Problem: find a state gain K so that y decays to zero at a desired rate

- apply $u = -Kx$, so the closed loop dynamic matrix is $A - BK$
- design K such that $\lambda(A - BK)$ is stable and lies in a desired region

Asymptotic tracking problem

design an overall system so that $y \to r$ as $t \to \infty$



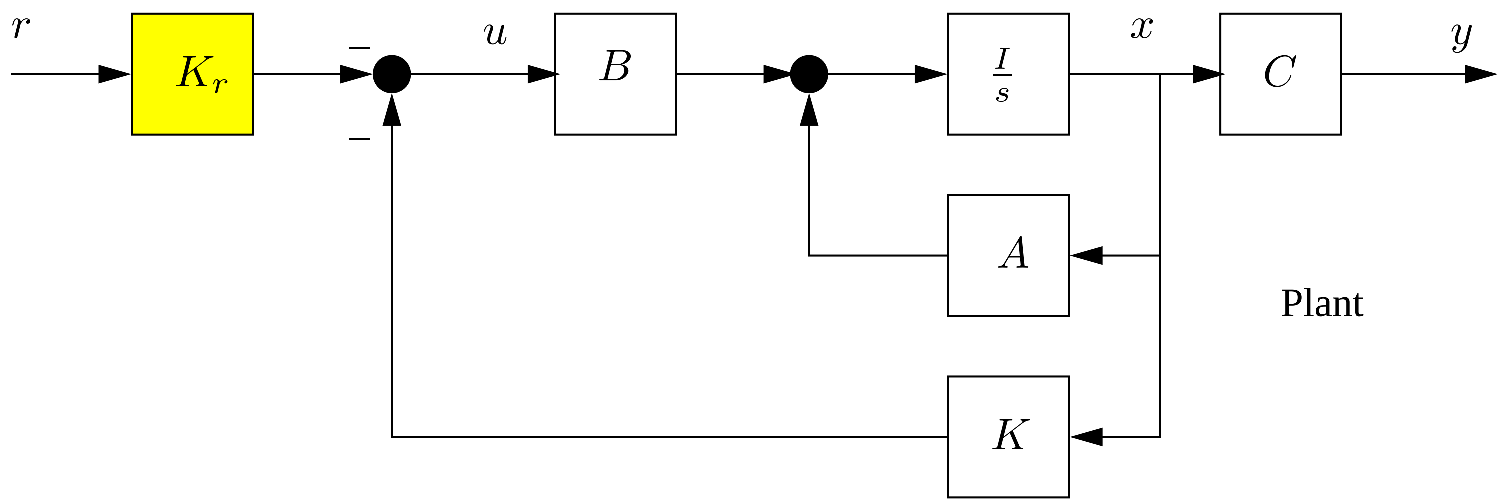
Left: a constant reference *Right:* a non-constant reference

- if $r(t) = 0$ then the tracking problem reduces to a regulator problem
- tracking a nonconstant reference is called a **servomechanism** problem

for a tracking problem, we use

$$u = -K_r r(t) - Kx(t)$$

K_r is a **feedforward gain** and K is a **feedback gain**



thus we have

$$\dot{x} = (A - BK)x + BK_r r$$

the transfer function from r to y is

$$\frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1}BK_r$$

assume (A, B) is controllable and $CA^{-1}B \neq 0$

- choose K such that $(A - BK)$ is stable
- choose K_r to make the DC gain from r to y:

$$-C(A - BK)^{-1}BK_r$$

  equal 1

then the closed-loop system can asymptotically track any **step** reference

Robust tracking and disturbance rejection

a *constant* disturbance w with unknown maginitude in the model

$$\dot{x} = Ax + Bu + Bw, \quad y = Cx$$

Objective: under a presence of

- disturbance w
- plant parameter variations (system uncertainties)

design u such that y asymptotically tracks any step reference

Idea: add an integrator to the system

Integrator: $\dot{z} = r - y$

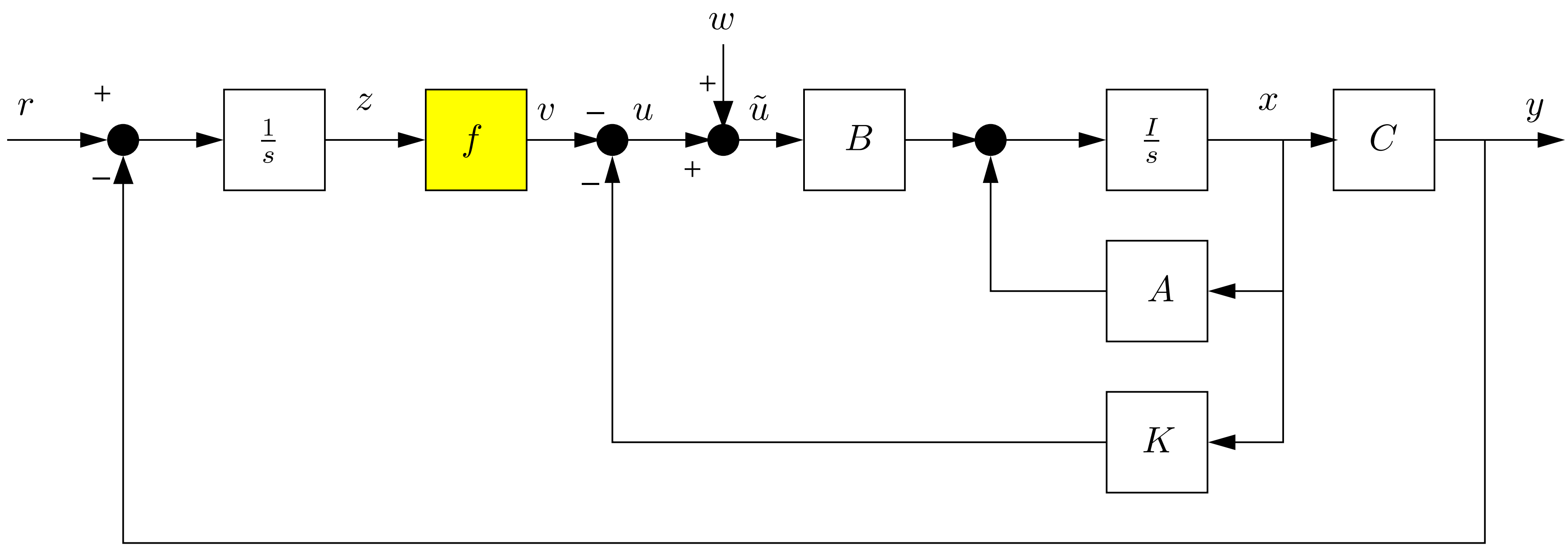
the state-space equation of the augmented system is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} w \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}$$

Fact: if (A, B) is controllable and $C(sI - A)^{-1}B$ has no zero at the origin ($CA^{-1}B \neq 0$) then

$$\left(\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix} \right) \quad \text{is controllable}$$

Control input: $u = -Kx - fz$

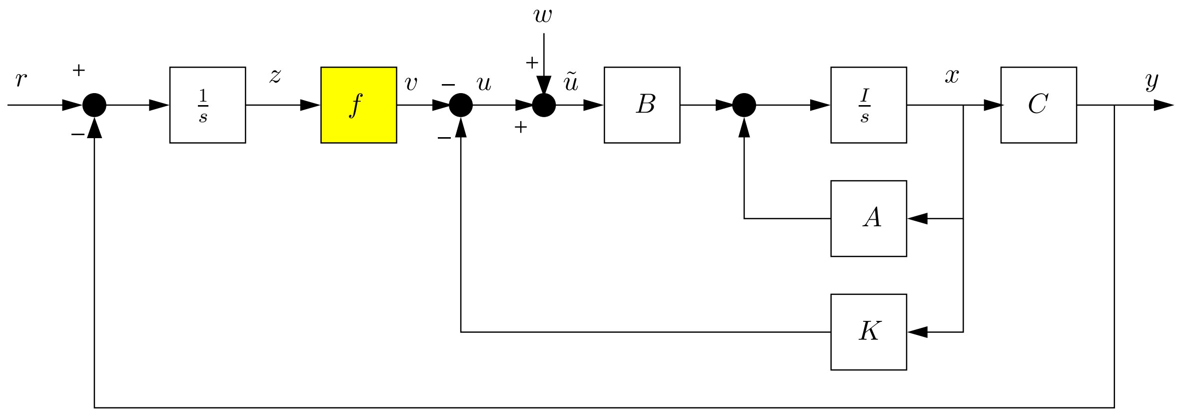


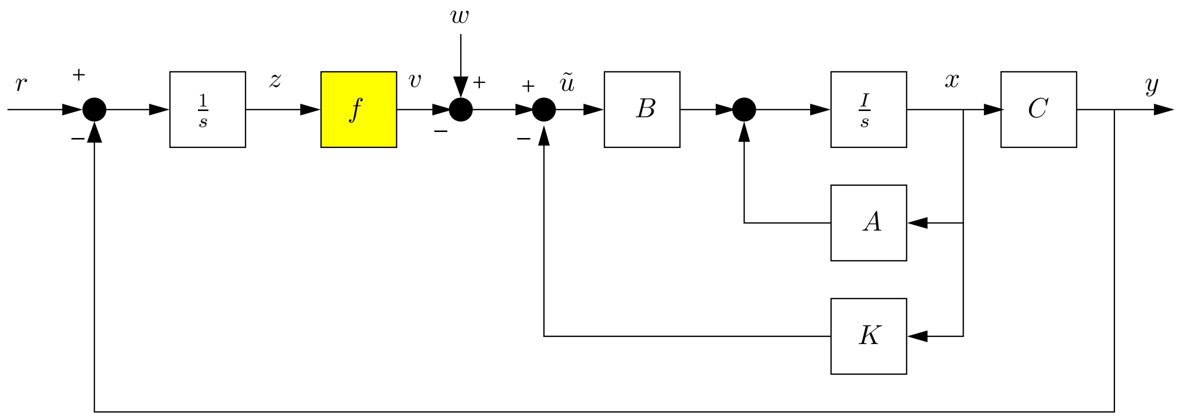
$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A - BK & -Bf\\ -C & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} r + \begin{bmatrix}B\\ 0\end{bmatrix} w, \quad y = \begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}$$

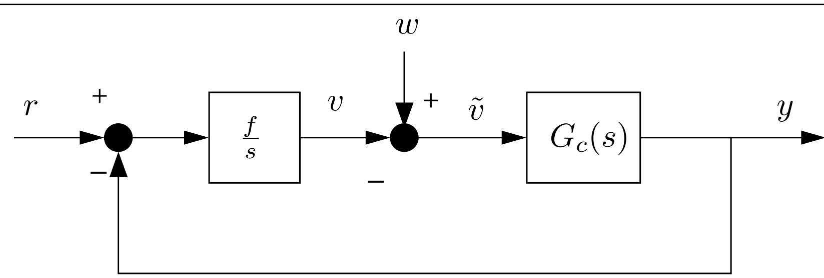
hence, under the conditions (A, B) controllable and $CA^{-1}B \neq 0$

the eigenvalues of the CL system can be freely assigned by $\begin{bmatrix}K & f\end{bmatrix}$

$$\tilde{u} = -v - Kx + w$$







$$G_c(s) = C(sI - A + BK)^{-1}B$$

closed-loop system (under state feedback only)

transfer function: $G_c(s) = C(sI - A + BK)^{-1}B = \frac{N_c(s)}{D_c(s)}$

note that

$$\begin{bmatrix} I & 0 \\ -C(sI - A + BK)^{-1} & 1 \end{bmatrix} \begin{bmatrix} sI - A + BK & Bf \\ C & s \end{bmatrix} = \begin{bmatrix} sI - A + BK & Bf \\ 0 & s - C(sI - A + BK)^{-1}Bf \end{bmatrix}$$

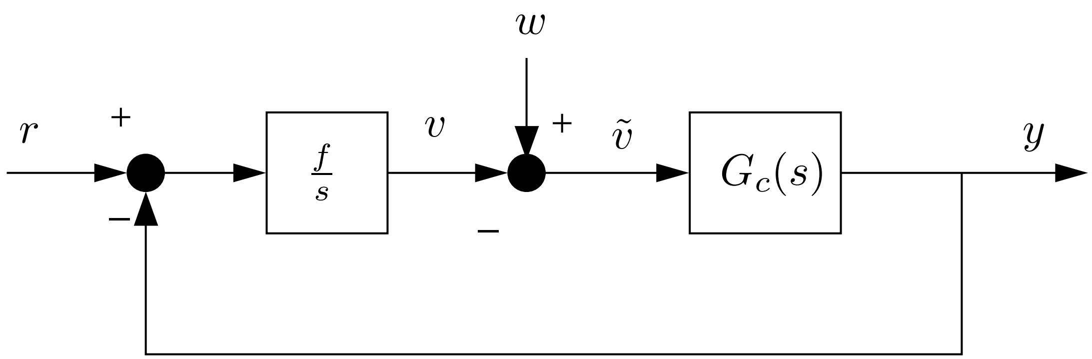
hence,

$$\det \begin{bmatrix} sI - A + BK & Bf \\ C & s \end{bmatrix} = \det(sI - A + BK) \cdot (s - C(sI - A + BK)^{-1}Bf)$$

closed-loop augmented system (with integrator)

characteristic equation: $\mathcal{X}_c(s) = sD_c(s) - fN_c(s)$

step disturbance rejection



by setting $r = 0$, the transfer function from w to y is

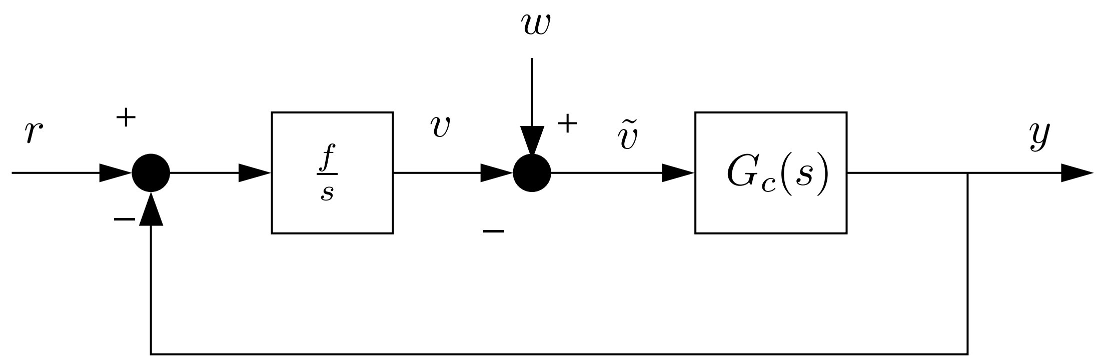
$$\frac{Y(s)}{W(s)} = \frac{sN_c(s)}{\mathcal{X}_c(s)}$$

hence, if $W(s) = 1/s$ then $Y(s) = \frac{N_c(s)}{\mathcal{X}_c(s)}$

if $\mathcal{X}_c(s)$ contains only stable poles (augmented CL system is stable)

the response of y due to w decays to zero regardless of magnitude of w

robust tracking



by setting $w = 0$, the transfer function from r to y is

$$\frac{Y(s)}{R(s)} = \frac{-fN_c(s)}{sD_c(s) - fN_c(s)} = \frac{-fN_c(s)}{\mathcal{X}_c(s)}$$

if $\mathcal{X}_c(s)$ has stable poles, the transfer function from r to y has DC gain=1

the response y tracks the reference even for the presence of parameter variations

Optimal state feedback

Idea:

- a drastic change in pole locations leads to a large feedback gain K
- a desired close-loop behavior is satisfied but use a large amount of input
- a trade-off between closed-loop performance and input energy should be considered in the control objective

Linear-quadratic optimal control

consider a *controllable* system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

LQR problem: find u that minimizes

$$\int_0^\infty x(t)^* Q x(t) + u(t)^* R u(t) \, dt$$

- $Q \succeq 0$ determine the cost of state performance
- $R \succ 0$ determine the cost of input energy
- u must stabilize the system, $i.e.$, we must have $x(t) \to 0$ as $t \to \infty$
- when $R = 0$, input u consists of impulsive inputs that *instantly* drive state to zero, so that optimal cost is zero
- if the system is stable and $Q = 0$ then optimal u is zero

Solution to LQR

introduce the **Algebraic Riccati equation (ARE)**

$$A^*P + PA - PBR^{-1}B^*P + Q = 0$$

- ARE is quadratic in P
- we are interested in a *positive definite* solution P

the solution of LQR problem is the optimal input u of the form:

$$u = -Kx$$

where the optimal feedback gain is

$$K = R^{-1}B^*P$$

the optimal cost function is $x_0^*Px_0$

Discrete-time LQR problem

consider a *controllable* discrete-time system

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

LQR problem: find u that minimizes

$$\sum_{k=0}^{\infty} x(k)^* Q x(k) + u(k)^* R u(k)$$

where $Q \succeq 0$ and $R \succ 0$

Solution to discrete-time LQR

introduce the **Discrete Algebraic Riccati equation (DARE)**

$$A^*PA - P - A^*PB(R + B^*PB)^{-1}B^*PA + Q = 0$$

- DARE is nonlinear in P
- we are interested in a positive solution P

the solution of LQR problem is the optimal input u of the form:

$$u = -Kx$$

where the optimal feedback gain is

$$K = R^{-1}B^*P$$

the optimal cost function is $x_0^*Px_0$

State observer

Idea:

- a state feedback requires the availability of *all* state variables
- if state variables cannot be acquired, we must design a *state estimator*

consider a state equation

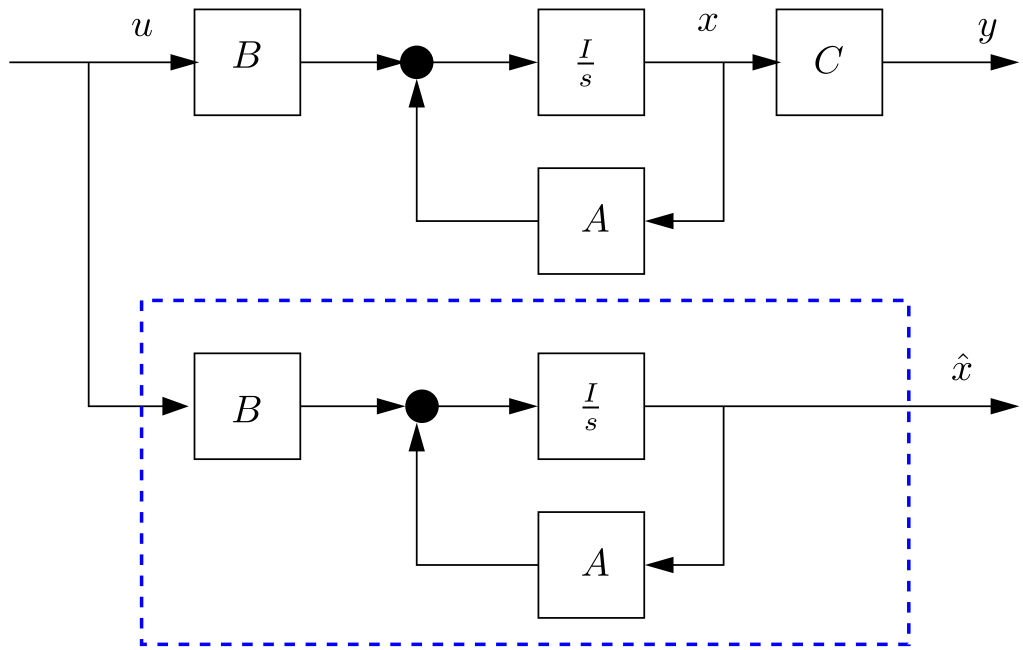
$$\dot{x} = Ax + Bu, \quad y = Cx$$

simple scheme: imitate the original system

$$\dot{\hat{x}} = A\hat{x} + Bu$$

- if (A, C) is observable, then $x(0)$ can be estimated
- initialize $\hat{x}$ by using $x(0)$ then $x(t) = \hat{x}(t)$ for all $t \geq T$ (for some T)

open-loop observer

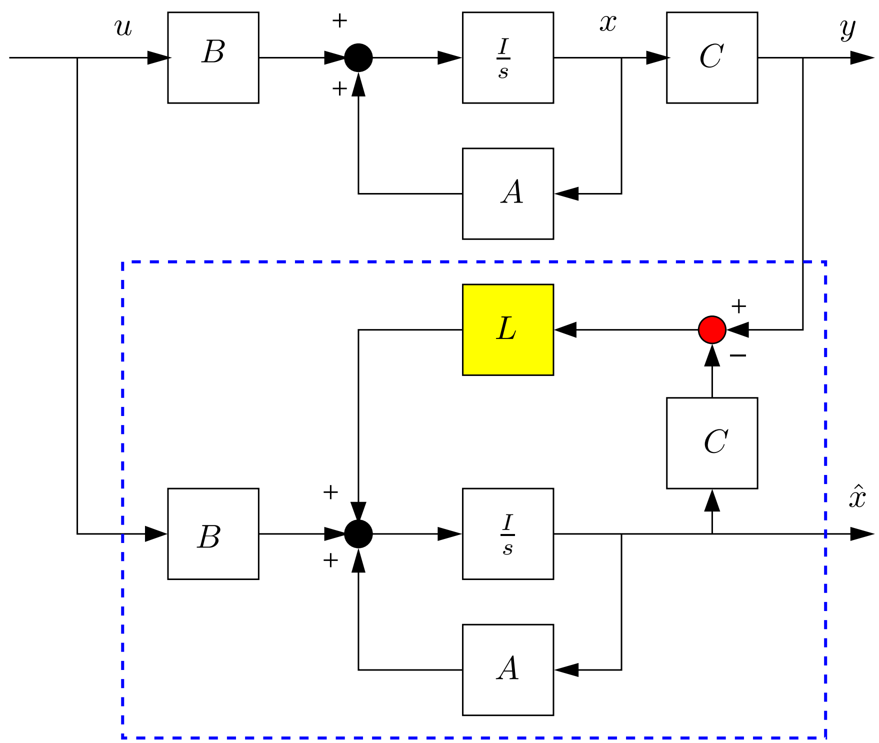


drawbacks:

- the initial state must be estimated each time we use the observer
- if A is unstable then the error between x and $\hat{x}$ grows with time

open-loop observer is not satisfactory in general

closed-loop state observer



we modify the state observer as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

add a correction term and design a proper gain L

observer gain: how to choose L ?

define the error between the actual state and the estimated state

$$e = x - \hat{x}$$

then the dynamic of e is

$$\begin{aligned}\dot{e} = \dot{x} - \dot{\hat{x}} &= Ax + Bu - (A - LC)\hat{x} - Bu - L(Cx) \\ &= (A - LC)x - (A - LC)\hat{x} = (A - LC)e\end{aligned}$$

- if $(A - LC)$ is stable then $e \to 0$, or $\hat{x}$ approach x eventually
- even if there is an initial large error $e(0)$, $e(t)$ still goes zero as $t \to \infty$ if $(A - LC)$ is stable
- no need to compute the initial estimate $\hat{x}(0)$ perfectly

Observer design

Fact: eigenvalues of $A - LC$ can be freely assigned iff (A, C) is observable

- change coordinate to the **observer canonical form**

$$\bar{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -a_{n-1} & 0 & 0 & & 1 & 0 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

- assume $\bar{L} = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}^T$, so

$$\det(sI - \bar{A} + \bar{L}\bar{C}) = s^n + (a_1 + l_1)s^{n-1} + \cdots + (a_n + l_n)$$

- choose the closed-loop poles arbitrarily by a suitable choice of $\bar{L}$
- transform $\bar{L}$ back to the new original coordinate

Remarks:

- observer design procedure can be obtained from the duality theorem:

$$(A, C) \text{ is observable if and only if } (A^*, C^*) \text{ is controllable}$$

- eigenvalues of $(A^* - C^*K)$ can be freely assigned by K if (A^*, C^*) controllable
- eigenvalues of $(A^* - C^*K)$ are the same as that of $(A - K^*C)$
- we can pick $L = K^*$
- designing an observer gain is equivalent to designing a state feedback gain for the dual system

Feedback from estimated states

when x is not available, we apply a state feedback from $\hat{x}$

$$u = r - K\hat{x}$$

this is called an **observer-based controller**

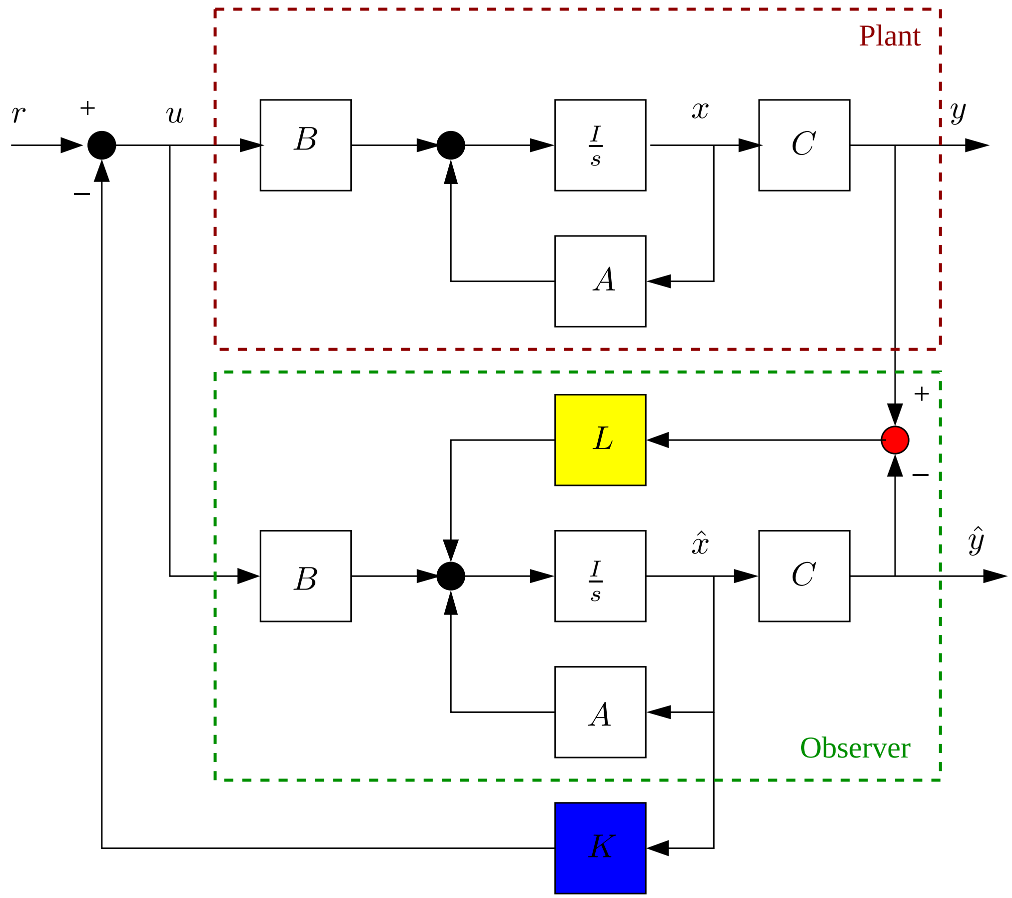
we have to answer the following questions

- is the closed-loop system stable ?
- using $u = -K\hat{x}$ gives the same set of eigenvalues as using $u = -Kx$?
- what is the effect of the observer on the transfer function from r to y ?

the state equation of the closed loop system

$$\dot{x} = Ax - BK\hat{x} + Br$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - LC)\hat{x} + B(r - K\hat{x}) + LCx$$



Separation property

the state equation in a vector form is

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

change of coordinate: define $e = x - \hat{x}$

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad \triangleq \quad T^{-1} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

hence, in the new coordinate the state equation is

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

- closed-loop eigenvalues are $\{\text{eig}(A - BK)\} \cup \{\text{eig}(A - LC)\}$
- designs of state feedback and observer can be done *independently*

Transfer function of the system with observer

the state equation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

is of the **uncontrollable** form

hence, the transfer function equals that of the reduced equation

$$\dot{x} = (A - BK)x + Br, \quad y = Cx$$

or the transfer function from r to y is

$$G(s) = C(sI - A + BK)^{-1}B$$

same as the transfer function of the orginal state feedback system *without* using an observer what are good choices of K and L ? no simple answer

some ideas:

- LQG control: K and L are chosen to optimize a quadratic objectives and we need to solve two decoupled Riccati equations
- $\mathcal{H}_\infty$ control: K and L are chosen to optimize an $\mathcal{L}$-induced norm of the closed-loop system. need to solve two coupled Riccati equations
- $\mathcal{L}_1$ control: K and L are chosen to optimize a peak-amplitude of regulated output. need to solve optimization problem (LP)
- multi-objectives, $e.g.$, mixed LQG/$\mathcal{H}_\infty$

Summary

- eigenvalues of $(A - BK)$ can be freely reassigned iff (A, B) is controllable
- optimal LQR control input is a constant state feedback computed via ARE
- feedback observer design is equivalent to state feedback design on the dual system
- observer-based controller combines observer and state-feedback designs

References

Lecture note on

Observer-based Controller Design, D. Banjerdpongchai, EE635, Chulalongkorn University

Chapter 8 in

C. Chen, *Linear System Theory and Design*, Oxford University Press, 2009