# 11. Generalized Method of Moments

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# Introduction

- method of moments (MM) estimators solves the sample moment conditions that correspond to the population moment conditions
- general methods of moments (GMM) estimators extends MM approach to accommodate the case when there are more moment conditions to solve than the number of parameters
- GMM estimator defines a class of estimators; using different population moment conditions gives different GMM estimators (just as different densities lead to different ML estimators)

GMM estimators are based on the analog principle that **population** moment conditions lead to **sample** moment conditions that can be used to estimate parameters suppose $y$ is i.i.d. with mean $\mu$, in population we have

$$\mathbf{E}[y - \mu] = 0$$

replacing the expectation by the average operator yields the corresponding sample moment

$$(1/N) \sum_{i=1}^{N} (y_i - \mu) = 0$$

solving for $\mu$ leads to the estimator $\hat{\mu}_{\mathrm{mm}} = (1/N) \sum_{i=1}^{N} y_i = \bar{y}$

the MM estimate of the population mean is the **sample mean**

# MM estimate in $\Gamma$ distribution

a Gamma distribution has the pdf

$$f(y) = \frac{1}{\Gamma(\alpha)}\beta^{\alpha} y^{\alpha-1} e^{-\beta y}, \quad y \geq 0.$$

with the known moment generating function

$$\mathbf{E}[Y^k] = \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}$$

and consider the first two moments and their sample estimate $m_1, m_2$

$$m_1 \approx \mathbf{E}[Y] = \alpha/\beta, \quad m_2 \approx \mathbf{E}[Y^2] = \frac{\alpha(\alpha+1)}{\beta^2}$$

from which we can solve that

$$\hat{\alpha} = \frac{m_1^2}{m_2 - m_1^2}, \quad \hat{\beta} = \frac{m_1}{m_2 - m_1^2}$$

# MM estimate in uniform distribution

consider $X \sim \mathcal{U}(a, b)$ and the first two moments

$$m_1 \approx \mathbf{E}[X] = (a+b)/2, \quad m_2 \approx \mathbf{E}[X^2] = (a^2 + ab + b^2)/3$$

use $a = 2m_1 - b$ and plug into the other equation to get

$$(b - m_1)^2 = 3(m_2 - m_1^2) \quad \Longrightarrow \quad b = m_1 \pm \sqrt{3(m_2 - m_1^2)}$$

choosing the root that makes $b > a$

$$\hat{a} = m_1 - \sqrt{3(m_2 - m_1^2)}, \quad \hat{b} = m_1 + \sqrt{3(m_2 - m_1^2)}$$

note: we can also choose $m_2 \approx \mathbf{var}[X] = (b-a)^2/12$

# Examples of GMM estimators

- linear regression
- nonlinear regression
- maximum likelihood
- instrumental variables regression

# Linear regression as an example of MM

consider the linear regression model: $y = x^T\beta + u$ where we assume $\mathbf{E}[u|x] = 0$

using the **law of iterated expectations**

$$\mathbf{E}[xu] = \mathbf{E}[\mathbf{E}[xu|x]] = \mathbf{E}[x\mathbf{E}[u|x]] = 0$$

hence, we obtain $\mathbf{E}[xu] = \mathbf{E}[x(y - x^T\beta)] = 0$

replacing $\mathbf{E}$ by the average operator gives the **sample moment** condition:

$$(1/N)\sum_{i=1}^{N} x_i(y_i - x_i^T\beta) = 0$$

this yields

$$\hat{\beta}_{\mathrm{mm}} = (\sum_i x_i x_i^T)^{-1} \sum_i x_i y_i$$

LS estimator is therefore just a special case of MM estimation

# Nonlinear regression as an example of MM

the nonlinear regression model with **additive error** is

$$y = g(x, \beta) + u$$

the assumption $\mathbf{E}[u|x] = 0$ implies that for any function $h(x)$ we have

$$\mathbf{E}[h(x)(y - g(x, \beta))] = 0$$

a particular choice is

$$h(x) = \nabla_\beta g(x, \beta)$$

that leads to the sample moment condition:

$$(1/N) \sum_{i=1}^{N} \nabla g(x_i, \beta)(y_i - g(x_i, \beta)) = 0$$

which is the first-order conditions for the NLS estimators

# Quasi-maximum likelihood as an example of MM

the quasi MLE $\hat{\theta}_{\mathrm{mle}}$ is defined to be the estimator that maximizes a log-likelihood function that is **misspecified**, as the result of specification of the wrong density

- let $f(y|\theta)$ denoted the **assumed** joint density of $y_1, \ldots, y_N$
- let $h(y)$ denoted the **true** density
- define the **Kullback-Leibler information criterion (KLIC)**

$$\mathrm{KL} = \mathbf{E}\left[\log \frac{h(y)}{f(y|\theta)}\right]$$

  where expectation is w.r.t. $h(y)$

- KL takes a minimum of $0$ when $\exists \theta^\star$ s.t. $h(y) = f(y|\theta^\star)$
- KL indicate greater ignorance about the true density

**definition:** the quasi-MLE minimizes KL, the distance between $h(y)$ and $f(y|\theta)$

but we can write KL as

$$\mathrm{KL} = \mathbf{E}[\log h(y)] - \mathbf{E}[\log f(y|\theta)]$$

hence, equivalently, the quasi-MLE estimate maximizes

$$\mathbf{E}[\log f(y|\theta)]$$

as $\mathbf{E}[h(y)]$ does not depend on $\theta$

**conclusion:** a local minimum of KL occurs if $\mathbf{E}[\nabla \log f(y|x,\theta)] = 0$

replacing by the sample moment conditions gives an estimator that solves

$$(1/N)\sum_{i=1}^{N} \nabla \log f(y_i|x_i,\theta) = 0$$

so a quasi-MLE can be motivated as an MM estimator

# IV regression as an example of MM

assume the existence of instrument $z$:

- $\mathbf{E}[u|z] = 0$ or that $\mathbf{E}[y - X\beta|z] = 0$
- $z$ are correlated with $x$

using law of iterated expectation, the population moment conditons are

$$\mathbf{E}[z(y - x^T\beta)] = 0$$

the MM estimator solves the sample moment condition

$$\frac{1}{N}\sum_{i=1}^{N} z_i(y_i - x_i^T\beta) = 0$$

- if $z$ has the same dimension as $x$ then the MM estimator is

$$\hat{\beta}_{\rm mm} = \left(\sum_i z_i x_i^T\right)^{-1} \sum_i z_i y_i$$

which is the linear IV estimator $\hat{\beta}_{\rm iv} = (Z^T X)^{-1} Z^T y$

- if $z$ has a higher dimension that $x$, then we choose $\beta$ to minimize

$$Q(\beta) = \left[\frac{1}{N}\sum_{i=1}^N z_i(y_i - x_i^T\beta)\right]^T W_N \left[\frac{1}{N}\sum_{i=1}^N z_i(y_i - x_i^T\beta)\right]$$

where $W_N$ is $p \times p$ if $z \in \mathbf{R}^p$

this choice is the **general method of moments estimator**

# Generalized Method of Moments

GMM defines a class of estimators where different choice of moment condition and weighting matrix lead to different GMM estimators, just as different choices of distribution lead to different ML estimators

- method of moments estimator
- definition of GMM estimator
- distribution of GMM estimator
- optimal GMM

# General form of MM estimators

assume there are $m$ moment conditions for $n$ parameters:

$$\mathbf{E}[h(w, \theta^\star)] = 0$$

- $\theta \in \mathbf{R}^n$ and $\theta^\star \in \mathbf{R}^n$ is the value of $\theta$ in the dgp
- $h$ is an $m \times 1$ vector-valued function
- $w$ includes all observables ($y, x$ or instrument $z$)

some examples of $h(w) = h(y, x, z, \theta)$

| **moment function** $h(\cdot)$ | **estimation method** |
|---|---|
| $y - \mu$ | method of moments for population mean |
| $x(y - x^T\beta)$ | ordinary least-squares regression |
| $z(y - x^T\beta)$ | instrumental variables regression |
| $\partial \log f(y\|x, \theta)/\partial\theta$ | maximum likelihood estimation |

# Definition of MM estimator

if $m = n$ then method of moments can be applied

replace the population moment by the sample moment

the **method of moments estimator** $\hat{\theta}_{\mathrm{mm}}$ is defined to the solution of

$$\frac{1}{N}\sum_{i=1}^{N} h(w_i, \hat{\theta}) = 0$$

this is the zero gradient condition of the minimization:

$$Q(\theta) = \left[\frac{1}{N}\sum_{i=1}^{N} h(w_i, \theta)\right]^T \left[\frac{1}{N}\sum_{i=1}^{N} h(w_i, \theta)\right]$$

**example 1:** for MM estimate in uniform distribution where $\theta = (a, b)$

$$h(x,\theta) \triangleq \begin{bmatrix} x - (a+b)/2 \\ x^2 - (a^2 + ab + b^2)/3 \end{bmatrix}$$

or we can choose

$$h(x,\theta) \triangleq \begin{bmatrix} x - (a+b)/2 \\ (x-\mu)^2 - (b-a)^2/12 \end{bmatrix} = \begin{bmatrix} x - (a+b)/2 \\ (x - (a+b)/2)^2 - (b-a)^2/12 \end{bmatrix}$$

**example 2:** for MM estimate in $\Gamma$ distribution where $\theta = (\alpha, \beta)$

- $\mathbf{E}[Y^k] = \Gamma(\alpha + k)/(\beta^k \Gamma(\alpha))$
- regularity condition $\mathbf{E}[\nabla_\theta \log f(y;\theta)] = 0$

$$\mathbf{E}[\nabla_\theta \log f(y;\theta)] = \begin{bmatrix} \log y - \psi(\alpha) + \log \beta \\ y - \alpha/\beta \end{bmatrix} = 0, \quad \psi(\alpha) = \frac{d \log \Gamma(\alpha)}{d\alpha}$$

- if $y$ is gamma then $1/y$ is inverse gamma distributed with $\mathbf{E}[1/Y] = \beta/(\alpha - 1)$

we can define any pair of $4$ components in $h$ to estimate $(\alpha, \beta)$

$$h(x,\theta) \triangleq \begin{bmatrix} x - \alpha/\beta \\ x^2 - \alpha(\alpha+1)/\beta^2 \\ \log x - \psi(\alpha) + \log \beta \\ 1/x - \beta/(\alpha - 1) \end{bmatrix}$$

two of all possible six choices are

$$h_1(x,\theta) = \begin{bmatrix} x - \alpha/\beta \\ x^2 - \alpha(\alpha+1)/\beta^2 \end{bmatrix}, \quad h_2(x,\theta) = \begin{bmatrix} \log x - \psi(\alpha) + \log \beta \\ x - \alpha/\beta \end{bmatrix}$$

- the first MM estimate can be readily (and cheaply) obtained
- the second MM estimate corresponds to the MLE estimate

# Definition of GMM estimators

the GMM estimator is based on $m$ conditions with $n$ parameters to be estimated

- if $m = n$ the model is said to be **just-identified** and MM estimator is used
- if $m > n$ the model is said to be **overidentified** and MM cannot be applied

originally $\hat{\theta}$ is chosen so that $(1/N)\sum_i h(w_i, \hat{\theta})$ is as close to zero as possible

the **GMM estimators** $\hat{\theta}_{\rm gmm}$ is instead defined to be the problem of minimizing

$$Q(\theta) = \left[\frac{1}{N}\sum_{i=1}^{N} h(w_i,\theta)\right]^T W_N \left[\frac{1}{N}\sum_{i=1}^{N} h(w_i,\theta)\right]$$

where $W \succ 0$, possibly stochastic but does not depend on $\theta$

# First-order condition for GMM estimators

differentiating $Q$ w.r.t. $\theta$ yields the first-order conditions:

$$\left[\frac{1}{N}\sum_{i=1}^{N}\frac{\partial h(w_i,\hat{\theta})}{\partial\theta}\right]^T W \left[\frac{1}{N}\sum_{i=1}^{N} h(w_i,\hat{\theta})\right] = 0$$

- the conditions are generally nonlinear in $\theta$; use numerical method to solve it
- diffferent choices of $W$ lead to different estimators with different variances
- the optimal choice of $W$ is provided

# Distribution of GMM estimator

assumptions:

1. the dgp imposes the moment condition: $\mathbf{E}[h(w, \theta^\star)] = 0$
2. $h(\cdot)$ satisfies $h(w, \beta) = h(w, \theta)$ iff $\beta = \theta$
3. the following $m \times n$ matrix exists and is finite with rank $n$:

$$A = \mathbf{plim}(1/N) \sum_{i=1}^{N} \frac{\partial h(w_i, \theta^\star)}{\partial \theta}$$

4. $W_N \xrightarrow{p} W$ where $W$ is finite positive definite
5. $(1/\sqrt{N}) \sum_{i=1}^{N} h(w_i, \theta^\star) \xrightarrow{d} \mathcal{N}(0, B)$ where

$$B = \mathbf{plim}(1/N) \sum_{i=1}^{N} \sum_{j=1}^{N} h(w_i, \theta^\star) h(w_j, \theta^\star)^T$$

then the **GMM estimator** $\hat{\theta}_{\rm gmm}$, defined to be the root of

$$\nabla_\theta Q(\theta) = 0$$

is consistent for $\theta^\star$ and

$$\sqrt{N}(\hat{\theta}_{\rm gmm} - \theta^\star) \xrightarrow{d} \mathcal{N}(0, (A^TWA)^{-1}(A^TWBWA)(A^TWA)^{-1})$$

**special case:**

- if the data are independent over $i$ then $B$ is simplified to

$$B = \textbf{plim}\, \frac{1}{N}\sum_{i=1}^{N} h(w_i, \theta^\star)h(w_i, \theta^\star)^T$$

- in just-identified case $(m = n)$, the matrices $A, W, B$ are square and invertible, the result on MM becomes

$$\sqrt{N}(\hat{\theta}_{\rm mm} - \theta^\star) \xrightarrow{d} \mathcal{N}(0, A^{-1}BA^{-T})$$

# Estimated asymptotic covariance

we use consistent estimates of $A, B$:

- estimate of $A$: replace $\theta^\star$ by $\hat{\theta}$

$$\hat{A} = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial h(w_i, \hat{\theta})}{\partial \theta}$$

- estimate of $B$: consider when data are independent over $i$

$$B = \frac{1}{N}\sum_{i=1}^{N} h(w_i, \hat{\theta}) h(w_i, \hat{\theta})^T$$

GMM estimator is asymptotically normally distributed with mean $\theta^\star$ and estimated covariance is

$$\widehat{\mathbf{Avar}}(\hat{\theta}_{\mathrm{gmm}}) = (1/N)(\hat{A}^T W_N \hat{A})^{-1} \hat{A}^T W_N \hat{B} W_N \hat{A} (\hat{A}^T W_N \hat{A})^{-1}$$

# Example on MM estimate of Gamma distribution

refer to the choice of $h$ on page 11-17

$$h(x,\theta) = \begin{bmatrix} \log x - \psi(\alpha) + \log \beta \\ x - \alpha/\beta \end{bmatrix}$$

we can estimate $A$ and $B$

$$\hat{A} = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial h(x_i, \hat{\theta})}{\partial \theta} = \begin{bmatrix} -\psi'(\hat{\alpha}) & 1/\hat{\beta} \\ -1/\hat{\beta} & \hat{\alpha}/\hat{\beta}^2 \end{bmatrix}$$

$$\hat{B} = \frac{1}{N}\sum_{i=1}^{N} h(x_i, \hat{\theta}) h(x_i, \hat{\theta})^T$$

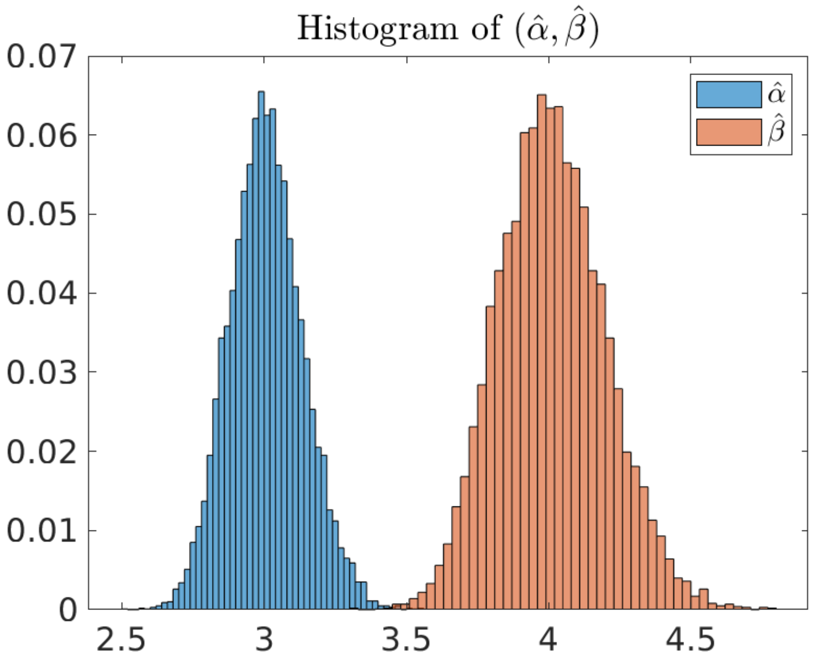
$$\widehat{\mathbf{Avar}}(\hat{\theta}_{\text{gmm}}) = (1/N)\hat{A}^{-1}\hat{B}\hat{A}^{-T}$$

by assuming that data are independent over $i$

# Simulation example

settings: the true parameter is $(\alpha, \beta) = (3, 4)$

- compute $\widehat{\mathbf{Avar}}(\hat{\theta}_{\mathrm{gmm}})$ computed on one data set with $N = 1000$
- compute $\hat{\theta}$ from $10,000$ data sets and check histogram/sample covariance




```
estimate_asymp_cov_theta =

    0.0154    0.0206
    0.0206    0.0328

sample_cov =
    0.0168    0.0224
    0.0224    0.0354
```

- the estimate of covariance (based on one data set) is similar to the sample covariance
- histograms approach a normal distribution

# References

Chapter 6 in

A.C. Cameron and P.K. Trivedi, *Microeconometircs: Methods and Applications*, Cambridge, 2005

Chapter 13 in

W.H. Greene, *Econometric Analysis*, Prentice Hall, 2008