# Senior Project Proposal 2102490 Year 2017

# A State-space model estimation of EEG signals using subspace identification


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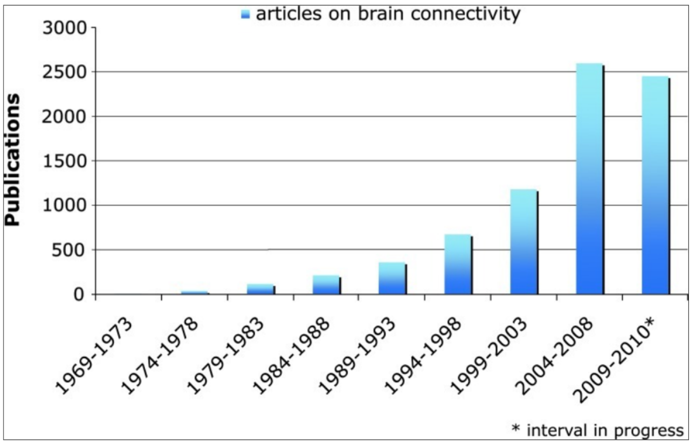



Figure 1: Number of publications in the PubMed database using the search term in "5-year increments". [1]

# 1 Introduction

Nowadays, there has been a growing interest in learning brain connectivity. According to number of brain connectivity publications indexed by PubMed (`https://www.ncbi.nlm.nih.gov/pmc/`), the number of publications are likely to be an exponential growth since 1969 [1]. Because the measure with associated signal processing are probably bring relevant information about the activity from activated network and also disrupted network that associated with tumors [2]. There are many methods to analyze how a group of neurons affects to the others, such as Dynamic causal modeling (DCM) and Granger causality (GC) [3]. One of widely-used method to analyze brain connectivity is Granger causality via Autoregressive (AR) model which is easy to explain.

$$y(t) = A_1 y(t-1) + A_2 y(t-2) + \cdots + A_p y(t-p) + e(t) \tag{1}$$

where $y$ is output signal, $e$ is measurement noise and $A_i$ is matrix explains relationship between signal in the past. However, time series data for actual EEG data may have Moving average (MA) component so that pure AR modeling may not be sufficient for EEG signals [4]. According to [5] , the result of Granger causality test on model (1) explain only causality relationship between output signal but the real objective is to find causality relations between sources in brain.

This project focuses on identifying EEG sources in state space model which can be described not only AR model but also Autoregressive Moving Average with Exogenous input (ARMAX) model. State space model has sufficient component of actual EEG data. With this advantage, we can analyze brain connectivity by using Granger causality on state space model so that we can learn causality relation between each source. In fact, we do not have information about parameters in state space model. The only information that we know is EEG time series data. For this reason, our study will also focus on subspace identification to identify system matrices for state space model. Subspace identification is a tool which is used for estimating state sequence and system matrices of model. This method is very useful because we can estimate all unknown state space variables and parameters with only prior knowledge (time series data).

# 2 Objectives

The objective of this study are the following

1. To estimate linear EEG model described by state space model using subspace method.
2. To learn brain connectivity for EEG signal by using Granger causality test on state space model.

# 3 Background

## 3.1 EEG Model

Located at the brain and spine, Central Nervous System (CNS) is the place where neural activities occurred. This happened by potential at gap between Axon and Dendrite called Synapse by stimulated to surround environment. An electroencephalogram (EEG) is one of tools to measure brain rhythms by measuring ionic current voltage fluctuations from electrodes placed on the scalp in special position [6] that specified using international 10-20 system. Each position is labeled with a letter and a number. The letter means area that electrode lied [7]. For example, F7 means node number 7 at Frontal lobe area.

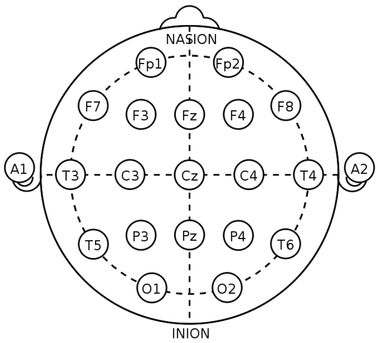



Figure 2: Electrode locations of International 10-20 system for EEG recording. The letters F,T,C,P and O stand for frontal, temporal, central, parietal, and occipital lobes, respectively.

From Fig.3, time series data is the data of 100 single-channel EEG segments of 4097 samples (23.6 seconds duration) dependence on recording region and brain state.

From the raw data, EEG are spontaneously non-stationary because statistical properties of the brain

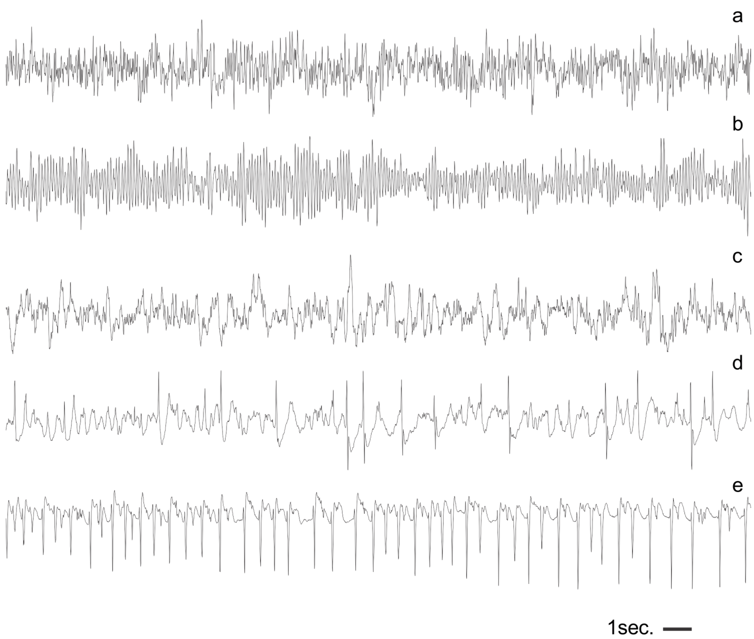



Figure 3: Raw data of EEG time series with awake state with eyes open (a) and eyes closed (b). The others were recorded during seizuring interval (c),(d) and during seizure activity (e) [8]

processes vary over time. Also, dynamical parameters of EEG are sensitive to time scales that involved in the process to get an insight in the working of brain [9]. By using EEG to analyze human brain activity, there are many mathematical model that describe EEG model. One can be generally described by linear Autoregressive (AR) model, which expressed as [10] :

$$x(t) = \sum_{k=1}^{p} A(k)x(t-k) + u(t) \tag{2a}$$

$$y(t) = Lx(t) + v(t) \tag{2b}$$

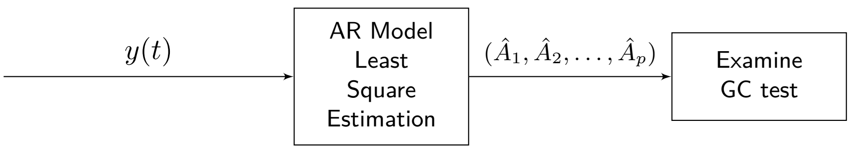
where $x \in \mathbf{R}^n$ is sample of brain source with $n$ nodes at time $t$ , $y \in \mathbf{R}^m$ is an EEG measurement (result show in terms of time series model from Figure 3) contains $m$ sources at time $t$ , $A_k \in \mathbf{R}^{n\times n}$ denotes parameters of past data , $L \in \mathbf{R}^{m\times n}$ means the lead field matrix that their magnitude are depend on head model, thickness of scalp and also volume conductor [11] , $u$ and $v$ are noise from source and noise from measurement, respectively and noise covariance matrix are given by :

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \mathbf{E}\left\{ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}^{\mathsf{T}} \right\} \tag{3}$$

### 3.2 Granger causality

Granger causality is a tool used for analyzing a brain connectivity. This tool is commonly expressed in terms of prediction error. The result of Granger causality for EEG indicates data from one part of the brain cause or does not cause to another part data. There are many approaches to examine GC Test. In this project, We focus on only two Granger causality test : GC test on AR model and GC test on state space model.

**Granger causality on AR model.** For linear AR model, Granger causality are performed after we estimate the system matrices of AR model shown as the following diagram.




For example, $x_1(t)$, $x_2(t)$ and $x_3(t)$ in AR model have relation shown as:

$$\begin{aligned} x_1(t) &= \sum_{k=1}^{p} a_k x_1(t-k) + \sum_{k=1}^{p} b_k x_2(t-k) + \sum_{k=1}^{p} c_k x_3(t-k) + \varepsilon_1(t) \\ x_2(t) &= \sum_{k=1}^{p} d_k x_1(t-k) + \sum_{k=1}^{p} g_k x_2(t-k) + \sum_{k=1}^{p} h_k x_3(t-k) + \varepsilon_2(t) \\ x_3(t) &= \sum_{k=1}^{p} m_k x_1(t-k) + \sum_{k=1}^{p} n_k x_2(t-k) + \sum_{k=1}^{p} r_k x_3(t-k) + \varepsilon_3(t) \end{aligned} \tag{4}$$

with covariance of noise as $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} = \mathrm{Cov}(\varepsilon)$.

Then, we assume that $x_2(t)$ is not a cause for $x_1(t)$ so the new model will be reduced and remain only $x_1(t)$ and $x_3(t)$

$$\begin{aligned} x_1(t) &= \sum_{k=1}^{p} a'_k x_1(t-k) + \sum_{k=1}^{p} c'_k x_3(t-k) + \varepsilon'_1(t) \\ x_3(t) &= \sum_{k=1}^{p} m'_k x_1(t-k) + \sum_{k=1}^{p} r'_k x_3(t-k) + \varepsilon'_2(t) \end{aligned} \tag{5}$$

with noise covariance of reduced model $\Sigma^R = \begin{bmatrix} \Sigma^R_{11} & \Sigma^R_{13} \\ \Sigma^R_{31} & \Sigma^R_{33} \end{bmatrix} = \mathrm{Cov}(\varepsilon')$.

After that, we examine if $x_2(t)$ has causality relation to $x_1(t)$ by determining log ratio of residual error of $x_1(t)$ for each model [12].

$$\mathcal{F}_{x_2 \to x_1 \mid x_3} = \log \frac{\Sigma^R_{11}}{\Sigma_{11}} \tag{6}$$

In general, $\Sigma^R_{11} > \Sigma_{11}$ because variance is minimized when data is added. From (6), if $\mathcal{F}_{x_2 \to x_1 \mid x_3} = 0$, it means $\Sigma^R_{11} = \Sigma_{11}$. Therefore, $x_2(t)$ is not cause $x_1(t)$. On the other hand, $x_2(t)$ cause to $x_1(t)$ when $\mathcal{F}_{x_2 \to x_1 \mid x_3} > 0$ because $x_1(t)$ in full model usually have more fitting than $x_1(t)$ in reduced model so that $\Sigma^R_{11}$ always more than $\Sigma_{11}$. Also, the result of GC test on AR model can be derived as $(A_k)_{ij} = 0$ and examine which $x_j$ is not a cause for $x_i$.

**Granger causality on state space model.** In case of state space model Granger causality test, state space equation is

$$z(t+1) = \mathcal{A}z(t) + w(t) \tag{7a}$$
$$y(t) = Cz(t) + v(t) \tag{7b}$$

In this Granger causality test, we examine if $y_j$ is a cause for $y_i$ by removing $y_j$ from the model. To remove $y_j$ from the model, we force $j^{th}$ column of $C$ from (7) be zero so that full model become reduced model. Then, determine residual error of both models. Finally, we determine log ratio of residual error of $x_i$ for each model [13].

$$\mathcal{F}_{y_j \to y_i \mid \text{All others } y} = \log \frac{|\Sigma^R_{ii}|}{|\Sigma_{ii}|} \tag{8}$$

where $|\Sigma^R_{ii}|$ and $|\Sigma_{ii}|$ are prediction error covariance of $x_i$ for reduced model and full model, respectively. Also, both $\Sigma^R_{ii}$ and $\Sigma_{ii}$ are calculated from optimal mean-squared error estimation which is derived by Kalman filter (further details in section 5.2). The result of Granger causality test : when $\mathcal{F}_{y_j \to y_i \mid \text{All others } y} > 0$ because $y_i(t)$ in full model usually have more fitting than $y_i(t)$ in reduced model and $\mathcal{F} = 0$ means $\Sigma^R_{ii} = \Sigma_{ii}$. Therefore, $y_j(t)$ does not cause $y_i(t)$.

# 4 Problem statement

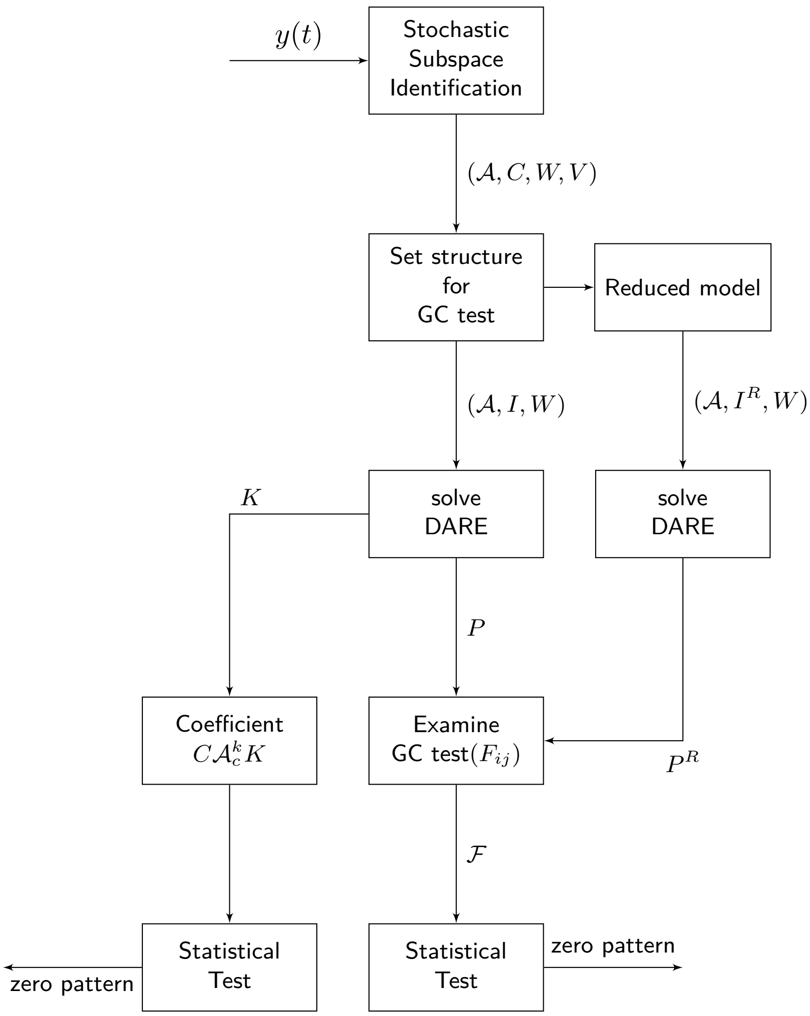
In this project, There are two main problems as follows.

- **Problem 1** : We estimate the system matrices of state space (7) with free parameters from time-series data by using subspace method because only measurement variable ($y$) are given and signal source ($z$) are not measured. The assumptions for parameters are that $(\mathcal{A}, C)$ are observable, $(\mathcal{A}, W)$ are controllable and noise covariance is positive definite.

- **Problem 2** : We examine Granger causality test from any estimated state space model (7) which are estimated by using subspace method from the problem 1. The test result from (8) ($\mathcal{F}$) is real source ($x$). The result from GC test can refer to brain connectivity.

# 5 Methodology

In this project, we estimate system matrices of state space model without structure from (7). We choose state space model to examine Granger causality. The scheme of model estimation is shown in the following diagram :




Our scheme starts with time-series data $y(t)$ which is the only data we know. $y(t)$ is generated based on Autoregressive ground truth model. From (7), we do not know sources : $x(t)$ and internal noise : $u(t)$ which is problem to compute system matrices since we want to estimate $\mathcal{A}$. No parameters are known. The variables and parameters describe in the following table.

Table 1: Variables and parameters in this project

| Variables | | Parameters | |
|---|---|---|---|
| Measured | Unknown | Known | Unknown |
| $y(t)$ | $x(t)$ | | $\mathcal{A}$ |
| | $u(t)$ | | $C$ |
| | | | $W, V$ |

We use subspace method to identify all system matrices ($\mathcal{A}$, $C$ and noise covariance). Then, set the estimated structure for Granger causality test by letting $C = I$ for full model and forced $j^{th}$ column of full model $C$ for reduced model, denoted as $C^R$ or $I^R$ for $C$ that remove $j^{th}$ column. There are two method to examine Granger causality test. First, solve discrete Riccati equation for both model that the solution is covariance of prediction error and compare the covariance of prediction of reduced model to full model by using Granger causality test (8), the result ($\mathcal{F}$ have to be verified by statistical test to make sure that the zero pattern of model is satisfy. Another method is to solve gain matrix from DARE ($K$) and examine coefficient $C\mathcal{A}_c^k K$. The result which verified by statistical test is also zero pattern of model.

## 5.1 Stochastic subspace method

We estimate sources and system matrices (in this case : $\mathcal{A}$, $C$,$W$,$V$) by using stochastic subspace method. The estimation process starts by estimating sources. Since, EEG linear model have no input so that the estimation will use stochastic subspace method. In this method we focus on estimate state sequence first. The process starts by dividing data by time to obtain past data and future data. Then, project the future output ($Y_f$) onto the past output ($Y_p$) space with zero initial state ($\hat{X}_0 = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \end{bmatrix}$) [14].

$$\mathcal{O}_i \triangleq Y_{i\,|\,2i-1}/Y_{0\,|\,i-1} = Y_f/Y_p \tag{9}$$

where $\mathcal{O}_i$ is the oblique projection and $Y_{0\,|\,i-1}$ is measurement data from $t = 0$ to $t = i-1$. After that, compute the state from single value decomposition (SVD) factorization.

$$\mathcal{O}_i = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_n V_1^T \tag{10}$$

Since $\mathcal{O}_i = \Gamma_i \hat{X}_i$ [15] and there are some non-singular matrix $T$ that $\Gamma_i = U_1 \Sigma_n^{1/2} T$ so that we obtain

$$\hat{X}_i = \Gamma_i^{\dagger} \mathcal{O}_i \tag{11}$$

Then, estimate system matrices in least-square sense by forming the equation

$$\begin{aligned} \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i\,|i} \end{bmatrix} &= \begin{bmatrix} A \\ C \end{bmatrix} \hat{X}_i + \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \\ \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} &= \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i\,|\,i} \end{bmatrix} \hat{X}_i^{\dagger} \end{aligned} \tag{12}$$

with noise covariance as

$$\begin{bmatrix} \hat{W} & \hat{S} \\ \hat{S}^T & \hat{V} \end{bmatrix} = (1/j) \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix} \begin{bmatrix} \rho_w \\ \rho_v \end{bmatrix}^T \tag{13}$$

## 5.2 Granger causality test on estimated state space model

This process happens after structured system matrices are solved. We use Granger causality test to examine brain connectivity of estimated model. In this process, Before the process starts, we assume that there is no measurement noise and $u(t)$ is also uncorrelated. In this process, we examine Granger causality test in two models : full model and reduced model. For full model, we assume $y(t)$ has the same dimension as $x(t)$. That means we force number of measurement sources equals to number of brain sources. This means we let $C = I$. To reduce the full model, we assume each $C_j = 0$ (column $j$ of $C$) which means we assume that value $x_j$ does not cause all others $x$ (since we assume that $y(t)_j = x(t)_j$ for all $j$) . Therefore, $y(t)$ is linear combination of all $x$ except for $x_j$.

Full model : $\quad z(t+1) = \mathcal{A}z(t) + w(t)\ ,\ y(t) = Cz(t) \qquad$ (14a)

Reduced model : $\quad z(t+1) = \mathcal{A}z(t) + w(t)\ ,\ y(t) = C^R z(t) \qquad$ (14b)

where $C = I$ and $C^R$ is reduced matrix that the $j^{th}$ column of $C$ is zero.
After that, we find estimation error covariance : $\Sigma = Cov(z - \hat{z}_{t\,|\,t-1})$ of both model. To obtain optimal prediction error covariance, we estimate $\hat{z}$ by using minimum mean square error because with this method, the error from noise is minimized. Therefore, we will get $\hat{x} = \mathbf{E}\{x_t \mid x_{t-1}^{-}\}$where $y_{t-1}^{-}$ is all output data from the past up to time $t-1$ and $\hat{x} = \mathbf{E}\{x_t^R \mid x_{t-1}^{R-}\}$ for reduced model. After this, we calculate estimation error covariance by using Kalman Filter [16] because of optimal method in linear model form :

$$\begin{aligned} \hat{x}_{t+1\,|\,t} &= \mathcal{A}\hat{x}_{t\,|\,t-1} + \mathcal{A}\Sigma_{t\,|\,t-1} C^T (C\Sigma_{t\,|\,t-1}C^T + R)^{-1}(y_t - C\hat{x}_{t\,|\,t-1}) \\ &= \mathcal{A}\hat{x}_{t\,|\,t-1} + K(y_t - \hat{y}_{t\,|\,t-1}) \end{aligned} \tag{15}$$

where $K = \mathcal{A}\Sigma_{t\,|\,t-1}C^{\intercal}(C\Sigma_{t\,|\,t-1}C^T + V)^{-1}$ is Kalman gain $K$ from (7) and $w_t$ can be expressed by $y_t - \hat{y}_{t\,|\,t-1}$ where $\hat{y}$ is estimated by MMSE ($\hat{y} = \mathbf{E}\{y_t \mid y_{t-1}^-\}$) and for reduced model we will get $\varepsilon^R$ from $y_t^R - \mathbf{E}\{y_t^R \mid y_{t-1}^{R-}\}$. From (15), time update gives a recursive solution. Therefore, we can express measurement and time update of $\Sigma$ as Riccati recursion [16].

$$\begin{aligned}\Sigma_{t+1\,|\,t} &= \mathcal{A}\Sigma_{t\,|\,t-1}\mathcal{A}^T + W - \mathcal{A}\Sigma_{t\,|\,t-1}C^T(C\Sigma_{t\,|\,t-1}C^T + V)^{-1}C\Sigma_{t\,|\,t-1}\mathcal{A}^T \\ &= \mathcal{A}\Sigma_{t\,|\,t-1}\mathcal{A}^T + W - \mathcal{A}\Sigma_{t\,|\,t-1}C^T(C\Sigma_{t\,|\,t-1}C^T)^{-1}C\Sigma_{t\,|\,t-1}\mathcal{A}^T \end{aligned} \tag{16}$$

(Assume that $V$ is zero)

From (16) , this equation is the optimal way to find state prediction error covariance [17]. However, we assume observation noise covariance is positive definite, $(\mathcal{A}, C)$ are observable and $(\mathcal{A}, W)$ are controllable so that we can solve steady state Kalman filter instead. The estimation of steady state Kalman filter satisfies Discrete Algebaric Riccati Equation (DARE) :

$$\Sigma = \mathcal{A}\Sigma\mathcal{A}^T + W - \mathcal{A}\Sigma C^T(C\Sigma C^T)^{-1}C\Sigma\mathcal{A}^T \tag{17}$$

There are two methods to examine Granger causality. The first method is to find log ratio of covarience of prediction error ($\Sigma$ from solving of Riccati equation). Then, we suggest to determine the time-domain Granger causality shown as : [13]

$$\mathcal{F}_{x_j \to x_i\,|\,\text{All others } x} = \log \frac{|\Sigma_{ii}^R|}{|\Sigma_{ii}|} \tag{18}$$

where $|\Sigma_{ii}^R|$ and $|\Sigma_{ii}|$ is estimation error covariance of $x_i$ for reduced model and full model, respectively. In general, $\Sigma_{ii}^R$ is usually larger than $\Sigma_{ii}$ because variance is minimized when data is added. If the result is zero, it means $|\Sigma_{ii}^R| = |\Sigma_{ii}|$. Therefore, $x_j$ does not affect $x_i$ conditioning to all others $x$. Otherwise, the value is always positive because reduced model is come up with more covariance magnitude.

After we solve (17) the solution of DARE remains only $W$ when we assume no measure noise and $u(t)$ is uncorrelated. (See Appendix 8.1). Another method to examine Granger causality is to find the coefficient $C_i\mathcal{A}_c^k K_j$ when $k = 0, 1, \ldots, p-1$. Denote $\mathcal{A}_c$ as state observer closed loop observer gain, the results of coefficient are $A_{k+1}$ for all $k$ which have same structure. Therefore, $\mathcal{A}_c$ yields the necessary and sufficient condition by the Cayley-Hamilton Theorem. (See Appendix 8.2)

# 6 Preliminary results

In this section, We performed experiments for GC test after we have studied subspace method for estimating state space model, GC test for describing brain connectivity and MATLAB codes for the proposed schemes. The first experiment is to show equivalence of GC tests on AR model and state space model. After that, we applied the result of the first experiment to verify estimated state space model by using subspace method in the second experiment. Both experiments are conducted from the same ground truth AR model from (2).

## 6.1 Equivalence of Granger causality tests

The object of this experiment is to show the equivalence of GC tests on AR model and state space model where ground truth model is AR model. Since we know that the result of GC test from AR model (6) can be derived as $(A_k)_{ij} = 0$ (that means $x_j(t)$ does not cause $x_i(t)$), the expected outcome of GC test on state space model based on ground truth AR model should be the same as the result from AR model. In this experiment, ground truth AR model is generated from MATLAB file : `gen_sparseAR.m` [18]. The process of this experiment starts with format state space model from ground truth AR model.

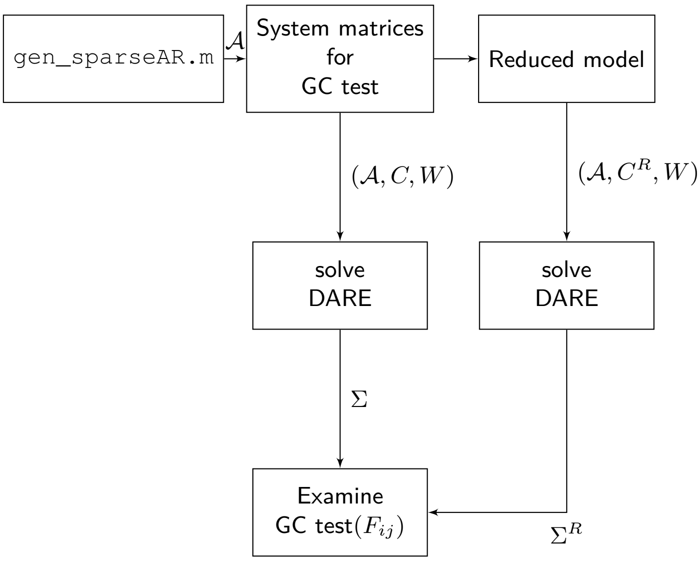
$$z(t+1) = \mathcal{A}z(t) + w(t) \tag{19a}$$

$$y(t) = Cz(t) + \varepsilon(t) \tag{19b}$$

where

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_p \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix}, \; C = \begin{bmatrix} L & 0 & \dots & 0 \end{bmatrix}, \; w(t) = \begin{bmatrix} u(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \varepsilon(t) = \begin{bmatrix} v(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then, we set state space system matrices for GC test by given $y(t) = x(t)$. Therefore, $C$ for GC test in this experiment becomes $C = \begin{bmatrix} I & 0 & \dots & 0 \end{bmatrix}$ for full model. For reduced model, the structure of $C^R$ for reduced model is $C$ for full model which $j^{th}$ column is removed. Moreover, we let measurement noise ($\varepsilon(t)$) to be zero and signal noise are uncorrelated for GC test. After that, we solve Riccati equation for both model so that we obtained residual error for both models. Finally, we examine GC test from (18). The process of this section shown as follow :




Given the structure of linear AR model shown as Fig. 4. The result from Ganger causality based on AR

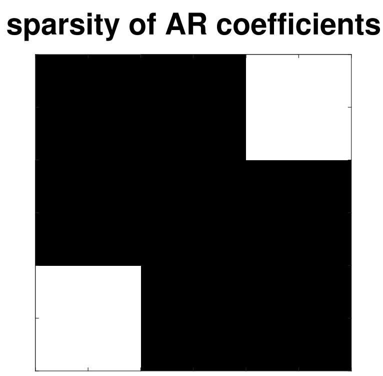



Figure 4: The sparsity for each $A_i$ that generated from `gen_sparseAR.m`

model from (6) is $(A_k)_{ij} = 0$. That means $x_j$ does not cause $x_i$. From figure 4, $A_{13}$ and $A_{31}$ is zero so that $x_3$ does not cause $x_1$ and $x_1$ does not cause $x_3$. The expected result of this experiment is the result of GC test on state space model that should be the same structure as GC test from AR model. From the MATLAB code, denoted $\mathcal{F}$ as GC test from ground truth AR model and $\mathcal{F}_r$ as GC test from state space.

```
>> F

F =

    0.5207    0.0016    0.0000
    0.2412    0.4073    0.1417
    0.0000    0.0282    0.1385

>> F_r

F_r =

    0.5207    0.0016    0.0000
    0.2412    0.4073    0.1417
    0.0000    0.0282    0.1385
```

It was found that the result of GC test from state space model based on ground truth model ($\mathcal{F}_r$) has the same structure as GC test from AR model ($\mathcal{F}$). The element of $\mathcal{F}$ and $\mathcal{F}_r$ shows the same value. This means Granger causality test on AR model is the special case for GC test on state space model since general state space have free parameters but state space model from AR has a fixed structure.

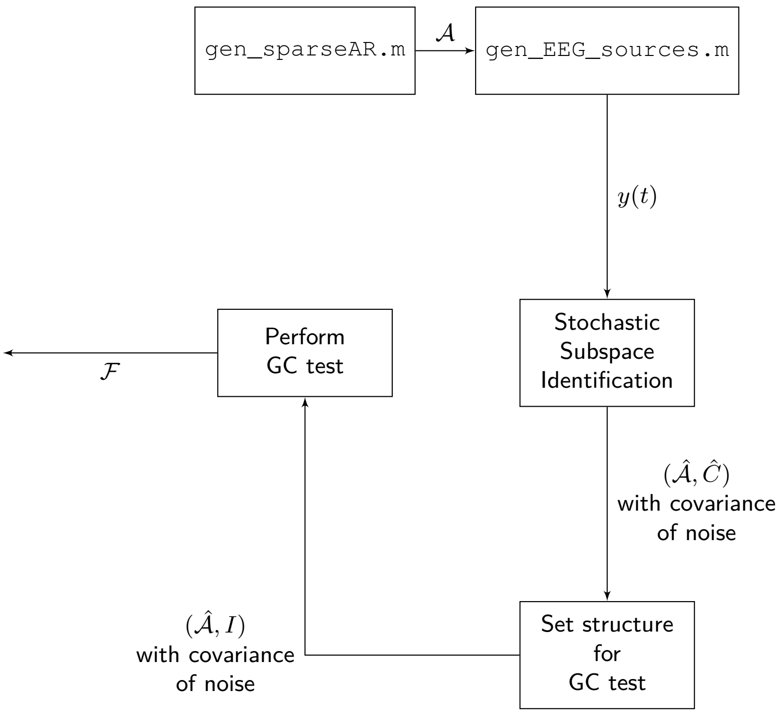
## 6.2 GC Test of estimated state space model

After we verified that GC test from state space models with ground truth AR model give the same result as GC test on AR model, we perform GC test for any estimated state space models in this experiment. The expected result of GC test on estimated state space model in this experiment is the same result as GC test on AR model. The estimated state space models were obtained by using subspace method with time series data from `gen_EEG_sources.m` based on system matrices from (19) which were generated from `gen_sparseAR.m` and lead field matrix $L$ is random with normal distribution. In procedure of subspace method, we use subspace identification toolbox in MATLAB called `n4sid` which is one of the subspace methods to determine system matrices and also noise variance in terms of innovation form [19] [20].

$$x(t+1) = Ax(t) + Ke(t) \tag{20a}$$

$$y(t) = Cx(t) + e(t) \tag{20b}$$

We assume that the dimension of estimated state space matrices are the same as dimension of all system matrices from (19). After subspace method were done, we obtained all system matrices ($\hat{A}, \hat{C}$ with covariance of noise). Then, we set state space model for GC test (14). Finally, we examine GC test from (18). The process of this section shown as follow :




We compared Granger causality test result between Autoregressive model ($\mathcal{F}$) and estimated state space model ($\mathcal{F}_{ss}$). For estimated state space model, we choose $np$ order because there is the same dimension as state space model with Autoregressive ground truth model.

```
>> F

F =

    0.5207    0.0016    0.0000
    0.2412    0.4073    0.1417
    0.0000    0.0282    0.1385

>> F_ss

F_ss =

    0.2162    0.2033    0.6843
    0.2532    3.1890    0.7549
    0.2575    0.9480    1.4804
```

The results showed that $\mathcal{F}$ from state space model that estimated from subspace identification are not the same value and not the same structure as AR model GC test result. The result implies contradiction to our hypothesis that $\mathcal{F}$ have to be same structure for AR model and estimated state space model based on ground truth AR model. The reason can be probably in the subspace identification, the fitting of estimated model ($\hat{y}$) is not high enough compared to actual data. The comparison of time series data are shown in figure 5

We generated 4097 data points in this experiment. As we look closer at the estimated time series data, the estimated data ($\hat{y}$) followed the actual data ($y$) until reach 40 datas in which $\hat{y}$ amplitude started to be unchanged follow to $y$

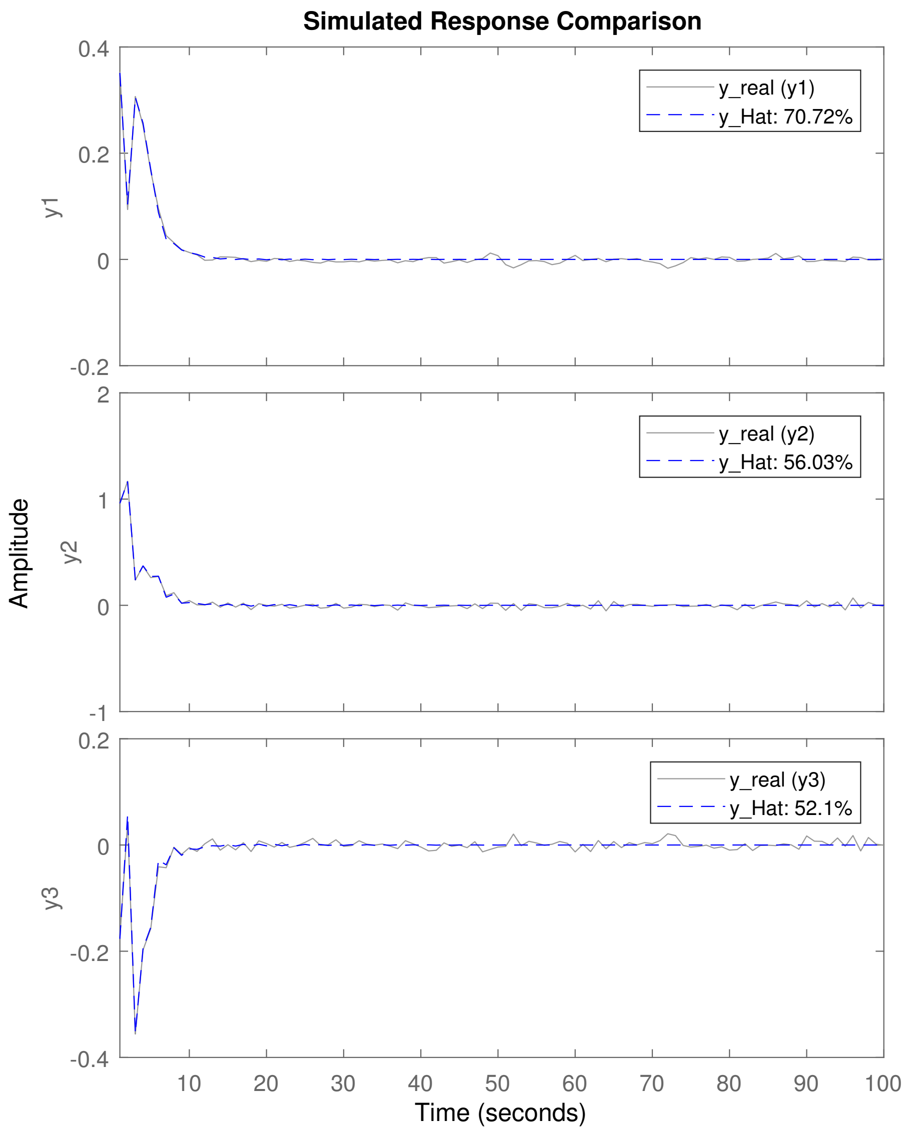



Figure 5: Comparison of simulated time series data with the data that we estimate from subspace identification at first 100 points

# 7 Project overview

## 7.1 Scope of work

The scope of this project are as follow :

1. We consider a comparison of Granger causality test
2. We consider linear time invariant model only and using estimation method based on least square.
3. Experimental results are mainly consist of
   (a) Show performance of Granger causality test from Autoregressive model compare to state space model on simulated data set.
   (b) Learn brain connectivity of real EEG data sets from healthy person and epilepsy person.

## 7.2 Expected outcomes

1. Schemes for estimating state space models that describe brain relationship of variables from time series.
2. MATLAB codes for the proposed scheme.
3. Comparison results of brain connectivity learned from two groups of EEG time series.

## 7.3 Project plans

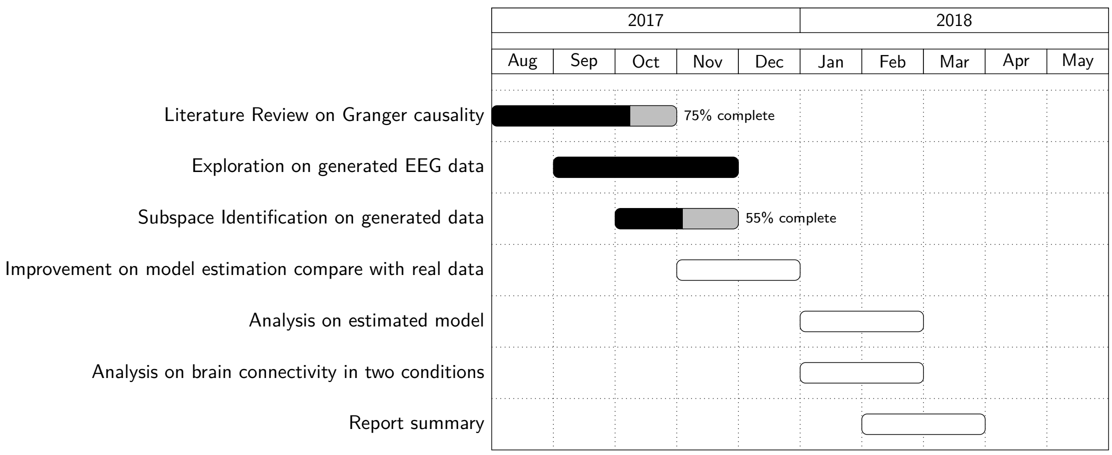



Figure 6: Gantt chart of the project

In the first semester, we reviewed on Granger causality on AR model and state space model. Moreover, we studied on MATLAB code that generate EEG data. Also, we studied subspace identification for estimating system matrices from simulated time series data. We decided to choose N4SID [19] as the method for subspace identification. However, we have to examine further about subspace identification toolbox MATLAB code because the result of an experiment did not satisfy GC test for AR model.

In the second semester, we will continue review on Granger causality. In this section, we will prove that with EEG data, Granger causality test for state space model are perform better than GC test for AR model. Moreover, after we study further about subspace identification, we will improve estimated model to analyze data from two groups of EEG time series : healthy person and person with epilepsy. Then, we analyze brain connectivity in two conditions.

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# 8 Appendices

## 8.1 Simplification of DARE applied to AR model

For AR model case, we examine GC test that $x_j(t)$ causes or does not cause $x_i(t)$ by comparing noise covariance of reduced model ($\Sigma_{ii}^R$ : Noise covariance when we remove $x_j$ from the model.) and full model by (6). When we determine GC test on state space model based on AR model (19), noise covariance can be calculated by using steady state Kalman filter that satisfies discrete Riccati equation (17). In this section, we demonstrate that the solution of discrete Riccati equation can be simplified to $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}$ We can simplify $\Sigma$ that we solve from into more similar form (17) by given $\left( W \in \mathbf{R}^{np \times np} \text{ and } W = \begin{bmatrix} W_1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix} \right)$

Given $\mathbb{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{p-1} \end{bmatrix}$ and $\Sigma \in \mathbf{R}^{np \times np}$ are in formation as : $\left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right]$ where $U_{ij} \in \mathbf{R}^{p \times p}$ denote the $(i,j)^{th}$ block of $U$

From

$$\Sigma = \mathcal{A}\Sigma\mathcal{A}^T + W - \mathcal{A}\Sigma C^T (C\Sigma C^T)^{-1} C\Sigma\mathcal{A}^T$$

we simplify in each term

$$\begin{aligned} \mathcal{A}\Sigma\mathcal{A}^T &= \left[\begin{array}{c|c} \mathbb{A} & A_p \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right] \left[\begin{array}{c|c} \mathbb{A}^T & I \\ \hline A_p^T & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbb{A}U\mathbb{A}^T + A_pV^T\mathbb{A}^T + \mathbb{A}VA_p^T + A_pRA_p^T & \mathbb{A}U + A_pV^T \\ \hline U\mathbb{A}^T + VA_p^T & U \\ \vdots & \end{array}\right] \end{aligned} \tag{21}$$

$$\begin{aligned} \mathcal{A}\Sigma C^T &= \left[\begin{array}{c|c} \mathbb{A} & A_p \\ \hline I & 0 \end{array}\right] \left[\begin{array}{c|c} U & V \\ \hline V^T & U \end{array}\right] \left[\begin{array}{c} I \\ \hline 0 \\ \\ 0 \end{array}\right] \\ &= \text{First block column of } \left[\begin{array}{c} \mathbb{A}U + A_pV^T \\ \hline U \end{array}\right] \\ &= \left[\begin{array}{c} \sum_{i=1}^{p} A_i\Sigma_{i,1} \\ \hline \Sigma_{11} \\ \Sigma_{21} \\ \\ \Sigma_{p-1,1} \end{array}\right] \end{aligned} \tag{22}$$

$$C\Sigma\mathcal{A}^T = (\mathcal{A}\Sigma C^T)^T = \left[\begin{array}{c|cccc} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T & \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \end{array}\right] \tag{23}$$

$$C\Sigma C^T = \Sigma_{11} \tag{24}$$

Then, combine all above terms (21), (22), (23) and (24) into DARE

$$
\left[\begin{array}{c|c} U & V \\ \hline V^T & R \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}U\mathbb{A}^T + A_pV^T\mathbb{A}^T + \mathbb{A}VA_p^T + A_pRA_p^T & \mathbb{A}U + A_pV^T \\ \hline U\mathbb{A}^T + VA_p^T & U \end{array}\right]
$$

$$
+ \begin{bmatrix} W_1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix} - \left[\begin{array}{c} \sum_{i=1}^{p} A_i\Sigma_{i,1} \\ \hline \Sigma_{11} \\ \Sigma_{21} \\ \vdots \\ \Sigma_{p-1,1} \end{array}\right] (\Sigma_{11})^{-1} \left[\begin{array}{c|cccc} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T & \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \end{array}\right] \tag{25}
$$

From (21)

$$
\begin{aligned}
U\mathbb{A}^T + VA_p^T &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1,p-1} \\ \Sigma_{21} & \ddots & & \Sigma_{2,p-1} \\ \vdots & & \ddots & \vdots \\ \Sigma_{p-1,1} & & & \Sigma_{p-1,p-1} \end{bmatrix} \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_{p-1}^T \end{bmatrix} + \begin{bmatrix} \Sigma_{1,p} \\ \Sigma_{2,p} \\ \vdots \\ \Sigma_{p-1,p} \end{bmatrix} A_p^T \\
&= \begin{bmatrix} \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\ \sum_{i=1}^{p} \Sigma_{2,i}A_i^T \\ \vdots \\ \sum_{i=1}^{p} \Sigma_{p-1,i}A_i^T \end{bmatrix}
\end{aligned} \tag{26}
$$

Determine $\Sigma_{21}$

$$
\begin{aligned}
\Sigma_{21} &= (\text{First row blocks of } U\mathbb{A}^T + VA_p^T) - \Sigma_{11}(\Sigma_{11}^{-1}) \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\
&= \sum_{i=1}^{p} \Sigma_{1,i}A_i^T - \sum_{i=1}^{p} \Sigma_{1,i}A_i^T = 0
\end{aligned}
$$

For the others $i$,$\Sigma_{2,i} = \Sigma_{1,i} - \Sigma_{11}(\Sigma_{1,1})^{-1}\Sigma_{1,i} = 0$. This means $\Sigma_{2,i} = 0$ for all $i = 1, 2, \dots, p$

Determine $\Sigma_{31}$

$$
\begin{aligned}
\Sigma_{31} &= (\text{Second row blocks of } U\mathbb{A}^T + VA_p^T) - \Sigma_{21}(\Sigma_{11}^{-1}) \sum_{i=1}^{p} \Sigma_{1,i}A_i^T \\
&= \sum_{i=1}^{p} \Sigma_{2,i}A_i^T = 0 \ (\Sigma_{2,i} = 0 \text{ for all } i)
\end{aligned}
$$

For the others $i$,$\Sigma_{3,i} = \Sigma_{2,i} - \Sigma_{21}(\Sigma_{11})^{-1}\Sigma_{2,i} = 0$ where $i = 2, 3, \dots, p$. This means $\Sigma_{3,i} = 0$ for all $i = 1, 2, \dots, p$
Consequently, $\Sigma_{i,j} = 0$ for all $i = 1, 2, \dots, p, j = 1, 2, \dots, p$ except for $\Sigma_{11}$

Determine $\Sigma_{11}$

$$
\begin{aligned}
\Sigma_{11} &= A_1\Sigma_{11}A_1^T + W_1 - A_1\Sigma_{11}(\Sigma_{11})^{-1}\Sigma_{1,1}A_1^T \\
&= W_1
\end{aligned}
$$

The result of Riccati equation remains only block $\Sigma_{11} = W_1$. Therefore, it satisfies that $\Sigma = W$

### 8.2 $C\mathcal{A}_c^k K$ coefficients

The results of GC test on AR model (6) can be derived as coefficient $(A_k)_{ij} = 0, \forall k$ which means $x_j(t)$ does not cause $x_i(t)$. Meanwhile, the results of GC test on state space model (8) can also be measured by $C\mathcal{A}_c^k K$ coefficient [13]. When $C_i\mathcal{A}^k K_j = 0, \forall k,\ i = 1, \ldots, n-1$, it means $x_j(t)$ does not cause $x_i(t)$ In this section, we showed that the coefficient from GC test on AR model have same structure to the coefficient from GC test on state space based on ground truth AR model.

$$\text{Coefficient of GC test on AR model :} \quad (A_k)_{ij} = 0\,, \forall k \tag{27a}$$

$$\text{Coefficient of GC test on state space model :} \quad C_i\mathcal{A}^k K_j = 0\,, \forall k,\ i = 1, \ldots, n-1 \tag{27b}$$

First, we determine $K$ from (15) based on ground truth AR model shown as :

$$\begin{aligned} K &= \mathcal{A}\Sigma C^T (C\Sigma C^T + V)^{-1} \\ &= \mathcal{A}\begin{bmatrix}\Sigma_{11}^T & \Sigma_{12}^T & \ldots & \Sigma_{1p}^T\end{bmatrix}^T \Sigma_{11}^{-1} \\ &\text{Because only } \Sigma_{11} \text{ is nonzero matrix} \\ &= \mathcal{A}\begin{bmatrix}I & 0 & \ldots & 0\end{bmatrix}^T \\ &= \begin{bmatrix}A_1^T & I & 0 & \ldots & 0\end{bmatrix}^T \end{aligned} \tag{28}$$

Given state observer closed loop observer gain $\mathcal{A}_c = \mathcal{A} - KC$. From (19) we have

$$\begin{aligned} \mathcal{A}_c &= \mathcal{A} - KC \\ &= \begin{bmatrix} 0 & A_2 & \ldots & A_p \\ 0 & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \ldots & I & 0 \end{bmatrix} \end{aligned} \tag{29}$$

Then, multiply by $C$ on the left hand side and $K$ on the right hand side :

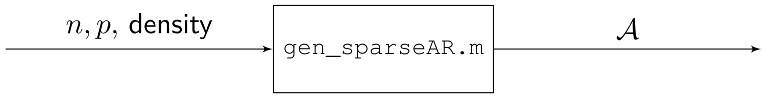
$$C\mathcal{A}_c^k K = \begin{bmatrix}I & 0 & \ldots & 0\end{bmatrix} \begin{bmatrix} 0 & A_2 & \ldots & A_p \\ 0 & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \ldots & I & 0 \end{bmatrix}^k \begin{bmatrix} A_1 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{30}$$

| | | |
|---|---|---|
| when $k = 0$ | $CK$ | $= A_1$ |
| when $k = 1$ | $C\mathcal{A}_c K$ | $= A_2$ |
| when $k = 2$ | $C\mathcal{A}_c^2 K$ | $= A_3$ |
| | $\vdots$ | $\vdots$ |
| when $k = p-1$ | $C\mathcal{A}_c^{p-1} K$ | $= A_p$ |

Because $A_1$ , $A_2$ , $\ldots$ , $A_p$ have the same structure so that if we assume $(A_1)_{ij} = 0$ that means $(\mathcal{A}_c)_{12} = 0$. Therefore, $\mathcal{A}_c = \mathcal{A} - KC$ yields the necessary and sufficient condition $C_i\mathcal{A}_c^k K_j = 0$ by the Cayley-Hamilton Theorem.

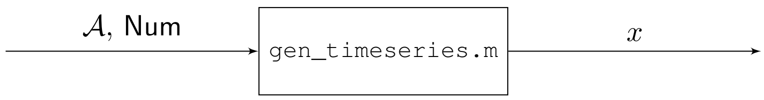
## 8.3 MATLAB functions in this project

- `gen_sparseAR.m` [18] : This function generates structure of each $A_i$ in AR model. To activate this function, state space dimension ($n$), number of AR order ($p$) and density is defined by user.




- `gen_EEG_sources.m` : This function generates time series data ($y$) from $\mathcal{A}$ that are generated from `gen_sparseAR.m` with lead field matrix $L$ are random with normal distribution


- `gen_timeseries.m` : This function generates EEG time series data that are calculated as the source from EEG later in `gen_EEG_sources.m`. "Num" is number of points for time series data.




- `GCTest.m` : This function calculate Riccati equation for both full model and reduced model. Then, calculate Granger causality function ($\mathcal{F}$) and also coefficient $C\mathcal{A}_c^k K$. The state space structure for GC test is required so that we have to set $C = I$ before activate this function.