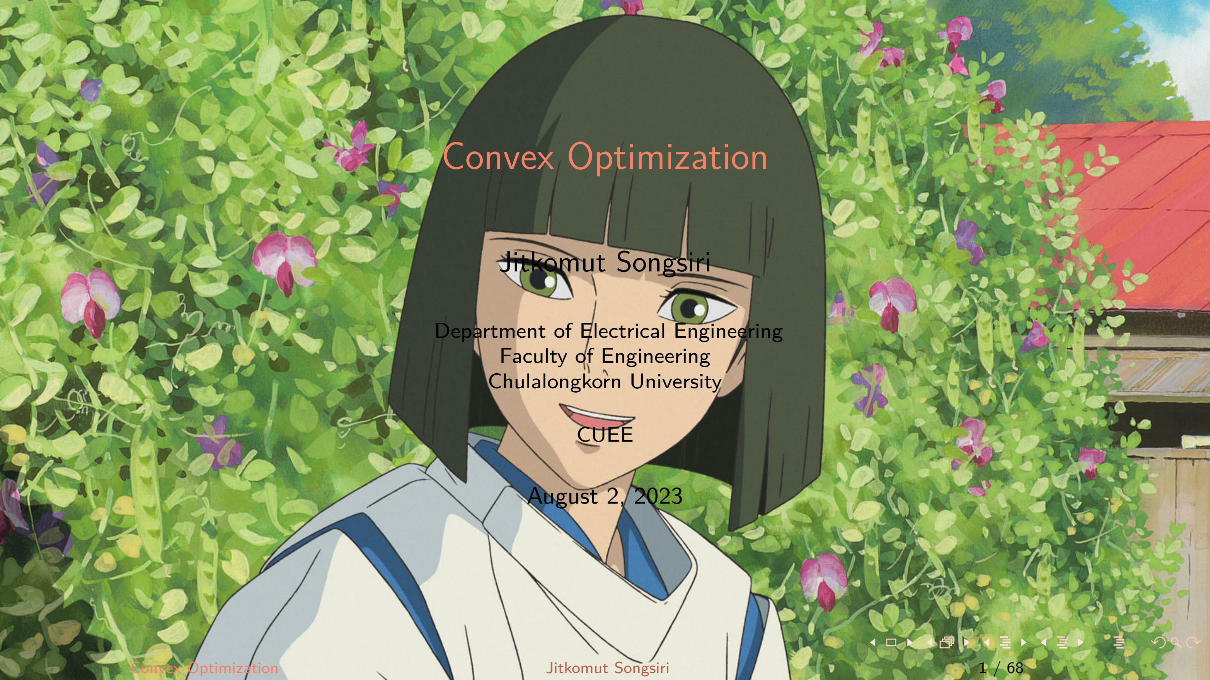

Convex Optimization

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Outline

1. General setting
2. Convex optimization
3. Problem Transformation
4. Structured convex problems
5. Optimality conditions
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General setting

Problem setting

(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \qquad \text{(P1)}$$

- $x = (x_1, \ldots, x_n)$: optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}, i = 1, \ldots, m$: inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}, i = 1, \ldots, p$: equality constraint functions

constraint set: $\mathcal{C} = \{x \in \mathbf{R}^n \mid f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, p\}$

domain of the problem: $\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i$

Optimal value

$$p^\star = \inf \{ f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ , i = 1, \ldots, p \}$$

- we say x is **feasible** if $x \in \mathbf{dom}\, f_0(x)$ and $x \in \mathcal{C}$
- $p^\star = \infty$ if the problem is **infeasible**
- $p^\star = -\infty$ if the problem is unbounded below
- a feasible x is called **optimal** if $f_0(x) = p^\star$; there can be many
- x is **locally optimal** if $\exists \epsilon > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & z \in \mathcal{C}, \quad \|z - x\|_2 \leq \epsilon \end{array}$$

in other words, a locally optimal point is the best solution in a neighborhood

Terminology

some equivalent definition/setting

- setting: another way of representing (P1)

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \in \mathcal{C} \qquad \text{(P2)}$$

- optimal point: we can also say $x^\star$ is a **global minimizer** of f_0 over $\mathcal{C}$

$$f_0(x) \geq f_0(x^\star) \quad \forall x \in \mathcal{C}$$

- local optimal point: we can also say $x^\star$ is a **local minimizer** of f_0 over $\mathcal{C}$

$$\exists \epsilon > 0 \quad \text{such that} \quad f_0(x) \geq f_0(x^\star) \quad \forall x \in \mathcal{C} \cap \|x - x^\star\| < \epsilon$$

- the standard form has an **implicit constraint**: $x \in \mathcal{D}$
- the constraint set $\mathcal{C}$ contains **explicit constraints**
- the problem is called **unconstrained** if it has no explicit constraints

Feasibility problem

a feasibility problem

$$\text{find} \quad x \quad \text{subject to} \quad x \in \mathcal{C}$$

can be considered as a special case of the general problem with $f_0(x) = 0$

$$\text{minimize} \quad 0 \quad \text{subject to} \quad x \in \mathcal{C}$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

examples: $\mathcal{C}_1$ has two-, $\mathcal{C}_2$ has infinitely many feasible points, while $\mathcal{C}_3$ is infeasible

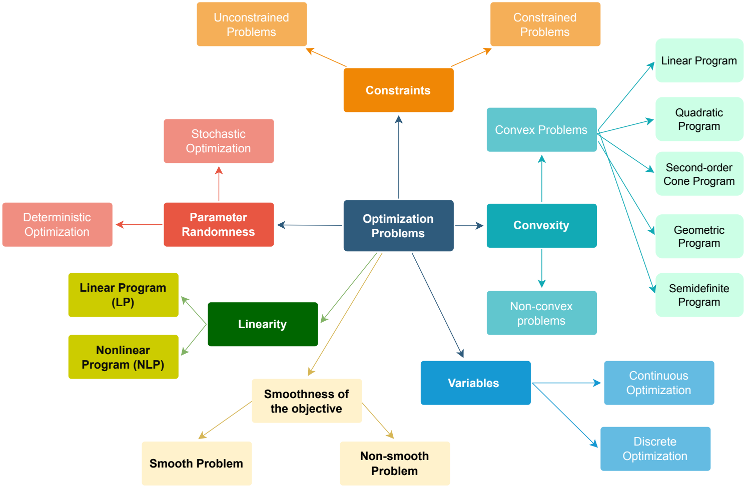
$$\begin{aligned}
\mathcal{C}_1 &= \{x \in \mathbf{R}^2 \mid (x_1 - 1)^2 + x_2^2 = 1, x_1 + x_2 = 1\} \\
\mathcal{C}_2 &= \{x \in \mathbf{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 1, x_1 + x_2 = 1\} \\
\mathcal{C}_3 &= \{x \in \mathbf{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 1, x_1 + x_2 = -3\}
\end{aligned}$$

Problem types

we can categorize optimization problems by

- **constraints**
  - unconstrained problem
  - constrained problems
- **variable types**
  - continuous optimization
  - discrete optimization
- **linearity of objective and constraints**
  - linear program
  - nonlinear program
- **convexity of objective and constraint set**
  - convex problem
  - non-convex problem
- **smoothness of the objective**
  - smooth problem
  - non-smooth problem
- **parameter randomness**
  - stochastic optimization
  - deterministic optimization

this course focuses on continuous and deterministic optimization



other specific problem types are integer programming, vector optimization.

Optimality of unconstrained problems

assumption: f is twice continuously differentiable (smooth objective)

- **1st-order necessary condition:**

$$\text{if } x^\star \text{ is a local minimizer of } f \text{ then } \nabla f(x^\star) = 0$$

- **2nd-order necessary condition:** if $x^\star$ is a local minimizer of f then $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star) \succeq 0$ (positive semidefinite)
- **2nd-order sufficient condition:** if $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star) \succ 0$ (pdf)

$$\text{then } x^\star \text{ is a strict local minimizer of } f$$

local minimizers can be distinguished from other stationary points by examining positive definiteness of $\nabla^2 f$

example: $f(x) = x^4$ has $x^\star = 0$ as a local minimizer; $\nabla^2 f(x^\star) = 0$ (hence, 2nd-order sufficient condition fails)

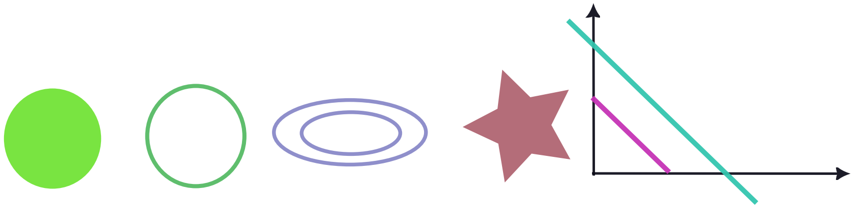
Convex optimization

Convex sets

a set $\mathcal{C}$ is said to be convex if for any $x, y \in \mathcal{C}$ we have

$$\theta x + (1-\theta)y \in \mathcal{C}, \quad \text{for all } 0 \leq \theta \leq 1$$

which of the following sets are convex ?



fact: an intersection of convex sets is convex (even infinitely many number of intersections)

Convex functions

convex function: $f : \mathbf{R}^n \to \mathbf{R}$ is convex if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

for all x, y in the domain of f and $0 \leq \theta \leq 1$

loosely speaking, f is convex if it has an upward shape

examples on $\mathbf{R}$:

- affine: $ax + b$ for any $a, b \in \mathbf{R}$
- exponential: e^{ax} for any $a \in \mathbf{R}$
- powers of absolute value: $|x|^p$ for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

Examples of convex functions on $\mathbf{R}^n$

- affine: $a^T x + b$
- norm functions: $\|x\|$
- norms of affine: $\|a^T x + b\|$
- quadratic: $x^T P x + q^T x$ when $P \succeq 0$
- negative entropy: $\sum_{i=1}^n x_i \log x_i$ on $\mathbf{R}^n_{++}$

fact: a set of inequality constraints described by convex functions is convex

$$\mathcal{C} = \{x \in \mathbf{R}^n \mid f_i(x) \leq 0, \; i = 1, 2, \ldots, m \}$$

is a convex set if all f_i's are convex functions

First- and second-order conditions of convex functions

suppose f is differentiable; then f is convex if and only if

$$\mathbf{dom}\, f \;\text{ is convex and }\; f(y) \geq f(x) + \nabla f(x)^T(y-x), \quad \forall x, y \in \mathbf{dom}\, f$$

- the first-order Taylor approximation of f is a **global underestimator** of f if and only if f is convex
- if $\nabla f(x) = 0$ then for all $y \in \mathbf{dom}\, f$, $f(y) \geq f(x)$,*i.e.*, x is a global minimizer of f

assume that $\nabla^2 f$ exists at each point in $\mathbf{dom}\, f$; then f is convex if and only if

$$\mathbf{dom}\, f \;\text{ is convex and }\; \nabla^2 f(x) \succeq 0, \;\; \forall x \in \mathbf{dom}\, f$$

f is convex if and only if its Hessian matrix is positive semidefinite

Convex programs

convex optimization problem is one of the form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
 & a_i^T x = b_i, \quad i = 1, \ldots, p
\end{array}
$$

where

- objective and constraint functions are **convex**
- equality constraint functions $h_i(x) = a_i^T x - b_i$ must be **affine**

result: an optimal solution of a convex program is a **global** minimizer

Properties of convex problems

convex problems are of interest due to some desirable properties

- many operations preserve convexity of a convex set
  - intersection
  - image (and inverse image) of affine mapping
  - image (and inverse image) of perspective mapping
- many operations preserve convexity of a convex function
  - non-negative weighted sum
  - composition with affine mapping, composition rules
  - pointwise maximum and supremum, minimization over one variable
  - perspective of a function
  - conjugate function (important role in duality theory)
- KKT conditions are *sufficient* and *necessary* for optimality

many optimization problems in engineering are convex programs

Linear program (LP)

a general linear program has the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where $G \in \mathbf{R}^{m \times n}$ and $A \in \mathbf{R}^{p \times n}$

example: minimize the cheapest diet that satisfies the nutritional requiremenets

- $x = (x_1, \ldots, x_n)$ is nonnegative quantity of n different foods
- each food has a cost of c_j; cost objective is $c^T x$
- one unit quantity of food j contains d_{ij} amount of nutrients i
- constraints are $Dx \succeq h$ and $x \succeq 0$

Geometrical interpretation

- hyperplane: solution set of a linear equation with coefficient vector $a \neq 0$

$$\{x \mid a^T x = b\}$$

- halfspace: solution set of a linear inequality with coefficient vector $a \neq 0$

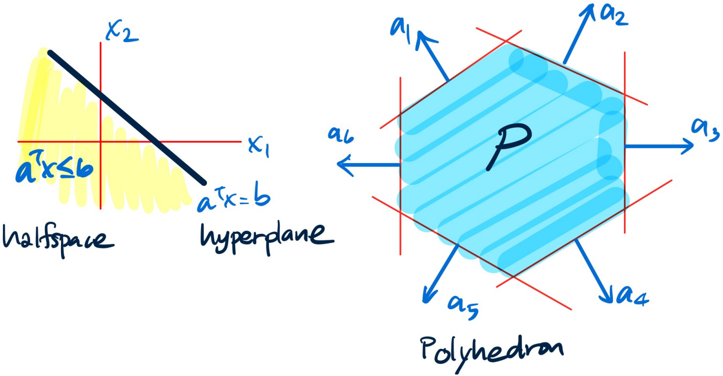
$$\{x \mid a^T x \leq b\}$$

  we say a is the **normal vector**

- polyhedron: solution set of a finite number of linear inequalities

$$\{x \mid a_1^T x \leq b_1, \;\; a_2^T x \leq b_2, \;\; \ldots, \;\; a_m^T x \leq b_m\} = \{x \mid Ax \leq b\}$$

  intersection of a finite number of halfspaces



extreme point of $\mathcal{C}$

a vector $x \in \mathcal{C}$ is an extreme point (or a vertex) if we cannot find $y, z \in \mathcal{C}$ both different from x and a scalar $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha)z$

Properties of LP

- another standard form: minimize $c^T x$ subject to $Ax = b$, $x \succeq 0$
- an LP may not have a solution (constraints are inconsistent or the feasible set is unbounded)
- we assume A is full row rank; if not, considering $Ax = b$
  - depending on A, the system could be inconsistent (hence, no extreme points), or
  - $Ax = b$ contains redundant equations, which can be removed
- if a standard LP has a finite optimal solution then

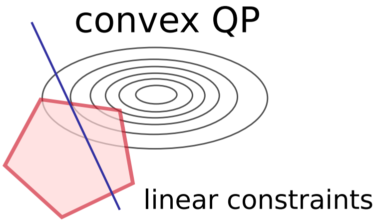
  a solution can always be chosen from among the vertices of the feasible set

  (called **basic feasible solutions**)
- the dual of an LP is also an LP
- solutions of some simple LPs can be analytically inspected

Quadratic program (QP)

a **quadratic program (QP)** is in the form

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array}$$



where $P \in \mathbf{S}^n, G \in \mathbf{R}^{m \times n}$ and $A \in \mathbf{R}^{p \times n}$

example: constrained least-squares

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & l \preceq x \preceq u \end{array}$$

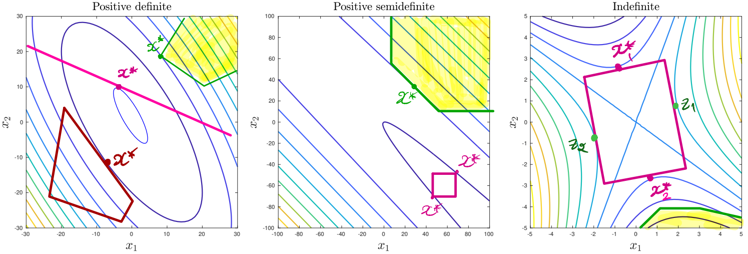
QP has **linear** constraints

Properties of QP

- an unconstrained QP is unbounded below if P is not positive definite
- an unconstrained QP has a unique solution: $x = -P^{-1}q$ when $P \succ 0$
- a QP is a convex problem if P is positive semidifinite definite
  - if $P \succeq 0$ then a local minimizer $x^\star$ is a global minimizer (by convexity)
  - if $P \succ 0$ then $x^\star$ is a *unique* global solution (by strictly convexity)
- the feasible set (polyhedron) may be empty (hence, the problem is infeasible)
- the feasible set can be unbounded (but if $P \succ 0$ it implies boundedness)
- solution of a QP may not be at a vertex
- the dual of a QP is also a QP

Contour of quadratic objective

consider three cases of P and different feasible sets



verify the location of the optimal solution for each constraint set

- left: a bounded set, a line, an unbounded feasible set
- middle: bounded and unbouded feasible sets, while f is unbounded below
- right: a bounded feasible set, while f is unbounded below and above

QCQP

a **quadratically constrained quadratic program (QCQP)** is in the form

$$
\begin{array}{ll}
\text{minimize} & (1/x)x^T P_0 x + q_0^T x \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b,
\end{array}
$$

where P_i's are positive semidefinite, $G \in \mathbf{R}^{m \times n}$ and $A \in \mathbf{R}^{p \times n}$

QCQP has both **linear and quadratic** constraints

Global optimum

consider a convex optimization problem: f is convex and $\mathcal{C}$ is a convex set

$$\underset{x}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad x \in \mathcal{C} \tag{1}$$

Theorem: (Ghaoui book, section 8.3.1)

- any locally optimal solution is also globally optimal
- the set $\mathcal{X}_{\text{opt}}$ of optimal points is convex

Proof

$$f_0(x) = \inf\{ f_0(z) \mid z \text{ is feasible}, \ \ \|z - x\|_2 \leq R \} \quad \text{for some } R > 0$$

Proof of global minimum

proof: let $x^\star$ be a local minimizer and $p^\star = f(x^\star)$

- for any $y \in \mathcal{C}$ then we can write $z \in \mathcal{C}$ as a convex sum: $z = \theta y + (1-\theta)x^\star$
- by the convexity of f

$$f(z) \leq \theta f(y) + (1-\theta) f(x^\star) \quad \Rightarrow \quad f(z) - f(x^\star) \leq \theta(f(y) - f(x^\star))$$

- since $x^\star$ is a local minimizer, LHS is non-negative if θ is small enough, then RHS is also non-negative
- we obtain $f(z) - f(x^\star) \geq 0$ for any $z \in \mathcal{C}$ – $x^\star$ is also global optimal
- the optimal set can be written as the $p^\star$-sublevel set

$$\mathcal{X}_{\rm opt} = \{x \in \mathcal{C} \mid f(x) \leq p^\star \}$$

- since a sublevel set of a convex function is convex, and f is convex, we have $\mathcal{X}_{\rm opt}$ is convex as claimed

Existence of solutions

Weierstrass extreme value theorem:

every continuous function $f : \mathbf{R}^n \to \mathbf{R}$ on a non-empty compact (closed and bounded) set attains its extreme values on that set

Theorem: sufficient condition for the existence

if $\mathcal{C} \subseteq \mathbf{dom}\, f$ is nonempty and compact and f is continuous on $\mathcal{C}$ then the problem (1) attains an optimal solution $x^\star$

note that this theorem is not applicable to an unconstrained convex problem (because $\mathcal{C} = \mathbf{R}^n$ which is not compact)

Coercive function

Definition: coercive functions

a function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be **coercive** if for any sequence $\{x_k\} \subset \int \mathbf{dom}\, f$ tending to the boundary of $\mathbf{dom}\, f$, it holds that the function value sequence $\{f(x_k)\}$ tends to $+\infty$ [1]

Lemma:

a continuous function with open domain is coercive *if and only if* all its sublevel sets $S_\alpha = \{x \mid f(x) \leq \alpha\}, \alpha \in \mathbf{R}$ are **compact**

[1] Ghaoui book, section 8.3.2

Existence of solutions (Coercive function)

Lemma: unconstrained optimization

if $\mathcal{C} = \mathbf{R}^n$ and f is continuous and **coercive**, then the convex optimization (1) attains an optimal solution $x^\star$

proof: take α that S_α is non-empty and follows the Weierstrass theorem

Lemma: constrained optimization

if $\mathcal{C} \subseteq \mathbf{dom}\, f$ is non-empty and closed, and f is continuous on $\mathcal{C}$ and **coercive**, then the convex problem (1) attains an optimal solution $x^\star$

Strictly convex function

a function f is said to be **strictly convex** if

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$

for all $x \neq y$ in the domain of f and $0 \leq \theta \leq 1$

- a strictly convex f satisfies the convexity condition with strict inequality
- $f(x) = a^T x + b$ is convex but not strictly convex
- intuitively, a convex function that has a 'flat' area is not strictly convex
- what about ϵ-insensitive loss function in SVR ?

Strongly convex

a function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be **strongly convex** on S if

$$\exists m > 0 \quad \text{such that} \quad f(x) - \frac{m}{2}\|x\|_2^2$$

is **convex** on S

related definition: if f is twice differentiable and

$$\nabla^2 f(x) \succeq mI, \quad \text{for all } x \in S$$

then f is said to be **strongly convex**

- example: $f(x) = x^T P x$ with $P \succ 0$
- a linear function $f(x) = a^T x + b$ is not strongly convex
- fact: ✎ a sum of convex and strongly convex functions is strongly convex

Strong convexity implies strict convexity

by convexity of $f(x) - \frac{m}{2}\|x\|^2$, that is

$$f(\theta x + (1-\theta)y) - \frac{m}{2}\|\theta x + (1-\theta)y\|_2^2 \le \theta f(x) + (1-\theta)f(y) - \frac{\theta m\|x\|^2}{2} - \frac{(1-\theta)m\|y\|^2}{2}$$

move the squared norm to the RHS and simplify

$$\begin{aligned} f(\theta x + (1-\theta)y) &\le \theta f(x) + (1-\theta)f(y) - \frac{m}{2}\theta(1-\theta)[\|x\|^2 - 2x^T y + \|y\|^2] \\ &\le \theta f(x) + (1-\theta)f(y) - \underbrace{\frac{m}{2}\theta(1-\theta)\|x-y\|_2^2}_{>0 \text{ for all } x \neq y} \end{aligned}$$

clearly, strong convexity implies strict convexity

Uniqueness of the optimal solution

Theorem:

if f is **strictly convex** in the problem (1), and $x^\star$ is an optimal solution, then $x^\star$ is the **unique** optimal solution

proof: let's prove by contradiction; let $x^\star$ be an optimal point and there exists another $y^\star \neq x^\star$ that is also optimal

- both $x^\star$ and $y^\star$ are feasible and $f(x^\star) = f(y^\star) = p^\star$
- let $\theta \in (0,1)$ and let $z = \theta x^\star + (1-\theta)y^\star$
- by convexity of $\mathcal{C}$, z must be also feasible
- by strict convexity of f,

$$f(z) < \theta f(x^\star) + (1-\theta)f(y^\star) = p^\star \quad \Rightarrow \quad z \text{ achieves a lower function value}$$

- this contradicts to the assumption that $x^\star$ is globally optimal

Strict convexity by regularization

let's add a quadratic term to a convex objective function

$$\tilde{f}(x) = f(x) + \gamma \|x - c\|_2^2$$

- clearly, $\|x - c\|_2^2$ is strongly convex
- a sum of convex and strongly convex is strongly convex
- hence, $\tilde{f}$ is strongly convex and also strictly convex
- minimizing $\tilde{f}$ over a convex set attains a unique optimal solution

Implications of strong convexity

obtain a **quadratic lower bound** on f (which is better)

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x), \quad \text{for } z \in [x,y]$$

$$f(y) \geq f(x) + \nabla f(x)^T (y-x) + \frac{m}{2}\|y-x\|_2^2, \quad \text{for } x, y \in S$$

- when $m = 0$, it reduces to the first-order condition for convexity
- strong convexity provides a higher lower bound than from convexity alone

Implications of strong convexity

obtain a **quadratic upper bound** for f on S

to see this, let $x \in \mathbf{dom}\, f$ and $y \in S = \{y | f(y) \leq f(x)\}$ (x is a fixed point)

$$y \in S \;\Rightarrow\; 0 \geq f(y) - f(x) \geq \underbrace{\nabla f(x)^T(y-x) + \frac{m}{2}\|y-x\|_2^2}_{A}$$

- a set of y such that $A \leq 0$ is the region inside a **bounded** ellipsoid
- then, the sublevel set S is contained in a bounded ellipsoid, so S is bounded
- when $\nabla^2 f$ is assumed to be continuous, it is bounded on a bounded set
- there exists $M > 0$ such that $\nabla^2 f(y) \preceq MI$ for all $y \in S$

$$f(y) \leq f(x) + \nabla f(x)^T(y-x) + \frac{M}{2}\|y-x\|_2^2, \quad \forall x, y \in S$$

Bounds on the optimality gap

for a strongly convex f and twice differentiable, it holds that

$$mI \preceq \nabla^2 f(x) \preceq MI, \quad \forall x \in S$$

and two inequalities for any points $x, y \in S$

$$\begin{aligned} f(y) &\geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} \|y - x\|_2^2 \\ f(y) &\leq f(x) + \nabla f(x)^T (y - x) + \frac{M}{2} \|y - x\|_2^2 \end{aligned}$$

Problem transformation

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one can be obtained from the solution of the other, and vice versa

examples: P1 and P2 are equivalent (but they are not the same)

minimize	$\|Ax - y\|_2$	(P1)	minimize	$\|Ax - y\|_2^2$	(P2)
maximize	$\frac{1}{\|Ax-y\|_2}$	(P1)	minimize	$\|Ax - y\|_2^2$	(P2)
maximize	$\lvert f(x)\rvert$	(P1)	maximize	$\log \lvert f(x)\rvert$	(P2)

using monotonically increasing property of squared and log functions

Transformation that yield equivalent problems

some transformations are useful for problem re-formulation

- eliminating equality constraints
- introducing slack variables
- epigraph form
- minimizing over some variables
- using indicator function to represent constraints

Eliminating equality constraints

the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \quad \Longleftrightarrow \quad x = Fz + x_0 \text{ for some } x_0$$

Example: eliminating equality constraints

equality constraint in the form of $Ax = b$ (non-trivial when A is fat)

$$\begin{array}{lll} \text{minimize} & \|Hx - y\|_2 & \text{(P1)} \\ \text{subject to} & x_1 + x_2 = 0 & \end{array} \qquad \begin{array}{lll} \text{minimize} & \|\tilde{H}x - y\|_2 & \text{(P2)} \\ \text{where} & \tilde{H} = \begin{bmatrix} h_1 - h_2 & h_3 & \cdots & h_n \end{bmatrix} & \end{array}$$

- find the nullspace of A and its basis vectors

$$\dim \mathcal{N}(A) = r \quad \Leftrightarrow \quad \exists F \in \mathbf{R}^{n \times r} \text{ such that } AF = 0 \text{ and } F \text{ is full column rank}$$

- find a particular solution of $Ax = b$, says x_0
- a general solutions to $Ax = b$ is expressed as $x = Fz + x_0$ for any z

Introducing slack variables

the problem

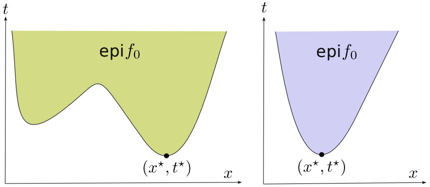
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) + s_i = 0, \quad i = 1, \dots, m \\ & s_i \geq 0, \quad i = 1, 2, \dots, m \end{array}$$

Epigraph form

the epigraph of a function f_0 is the area above the graph f_0



epi f_0 $= \{(x,t); | \; x \in \mathbf{dom} f_0, \;\; f_0(x) \leq t\,\}$

the standard problem is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, t) & t \\ \text{subject to} & f_0(x) - t \leq 0, \\ & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

we minimize t over the epigraph of f_0 (objective is now linear of (x, t))

Example: epigraph form

example 1: $\|z\|_\infty \leq t$ if and only if $|z_i| \leq t$ for all i

$$\text{minimize}_x \quad \|Ax - y\|_\infty \quad \text{(P1)}$$

$$\begin{array}{lll} \text{minimize}_{(x,t)} & t & \text{(P2)} \\ \text{subject to} & -t \leq a_i^T x - y_i \leq t & , i = 1, \ldots, m \end{array}$$

example 2: for a symmetric F, $\|F\|_2 \leq t$ if and only if $-tI \preceq F \preceq tI$

given symmetric matrices F_i for $i = 0, 1, \ldots, n$

$$\text{minimize}_x \quad \|F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n\|_2 \quad \text{(P1)}$$

$$\begin{array}{lll} \text{minimize}_{(x,t)} & t & \text{(P2)} \\ \text{subject to} & -tI \preceq F_0 + \sum_{i=1}^n x_i F_i \preceq tI & \end{array}$$

Minimizing over some variables

the problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \quad i = 1, \dots, m \end{array}$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

if the objective can be minimized over one variable easily, we can reduce the problem dimension

Example: minimizing over one variable

given $g_i : \mathbf{R}^n \to \mathbf{R}, y_i \in \mathbf{R}$ for $i = 1, \ldots, N$, consider the problem

$$\underset{x,d}{\text{minimize}} \quad -N \log \left[\frac{1}{d}\right] + \frac{1}{d} \sum_{i=1}^{N} (g_i(x) - y_i)^2$$

first, we can minimize over d by setting the gradient w.r.t. $1/d$ to zero

$$d = \frac{1}{N} \sum_{i=1}^{N} (g_i(x) - y_i)^2$$

the reduced problem is

$$\underset{x}{\text{minimize}} \quad \log \left[\frac{1}{N} \sum_{i=1}^{N} (g_i(x) - y_i)^2\right] \quad \Longleftrightarrow \quad \underset{x}{\text{minimize}} \quad \sum_{i=1}^{N} (g_i(x) - y_i)^2$$

Constraints expressed as indicator functions

introduce the **indicator function** associated with a set $\mathcal{C}$

$$I_{\mathcal{C}}(x) = \begin{cases} 0, & x \in \mathcal{C} \\ +\infty, & x \notin \mathcal{C} \end{cases}$$

the minimization of $f_0(x)$ subject to $x \in \mathcal{C}$ is equivalent to

$$\underset{x}{\text{minimize}} \quad f_0(x) + I_{\mathcal{C}}(x)$$

note that $I_{\mathcal{C}} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is extended-value function

we express the original constrained problem as an unconstrained problem usign $I_{\mathcal{C}}$

Structured convex problems

some structures that are amenable for parallel and distributed algorithms

- **separable sum**

$$\underset{x_1,\ldots,x_m}{\text{minimize}} \quad f(x) := \sum_{i=1}^{m} f_i(x_i)$$

  it is obvious that we can minimize over x_i independently

- **global consensus**

$$\underset{x}{\text{minimize}} \quad f(x) := \sum_{i=1}^{m} f_i(x)$$

  f_i is a local objective; x is the global variable

  consensus form: add a consensus constraint that makes all local x_i's agree

$$\underset{x_1,\ldots,x_m}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x) \quad \text{subject to} \quad x_1 = x_2 = \cdots = x_m$$

Structured convex problems

Structured convex problems

- **global exchange**

$$\underset{x}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x_i) \quad \text{subject to} \quad \sum_{i=1}^{m} x_i = 0$$

interpretation: x_i's are quantities of commodities exchanged among m agents

goal: minimize total social cost subject to the market clearing

- **allocation**

$$\underset{x}{\text{minimize}} \quad \sum_{i=1}^{m} f_i(x_i) \quad \text{subject to} \quad x_i \geq 0, \ \sum_{i=1}^{m} x_i = b$$

interpretation: x_i's are non-negative resources allocated to m activities

goal: minimize each activity cost while the total resource is limited to a budget

Distributed model fitting

a problem of fitting y using a linear model Ax using a loss function l

$$\underset{x}{\text{minimize}} \quad l(Ax - y) + r(x)$$

$l(Ax - y) = \sum_{i=1}^{N} l_i(a_i^T x - y_i)$ represents the model cost due to error $Ax - y$

r is a separable function representing **regularization**, *e.g.*, $\|\cdot\|_1, \|\cdot\|_2^2$

this is an example of global consensus

a common model parameter x that makes the model fits with *all* data samples

Nonsmooth optimization

a function is smooth if it is differentiable and the derivatives are continuous

- example: $f(x) = |x|$ is not smooth at $x = 0$
- example: $f(x) = \|x\|$ is not smooth at $x = 0$

a problem is called **nonsmooth** if the objective or constraints are nonsmooth functions

example: lasso problems

$$\text{minimize} \quad \|Ax - b\|_2 + \gamma\|x\|_1$$

then the methods relying on the gradient should be carefully revisited

Scalarized multi-objective optimization

a common form of multi-objective problem: for a given $\gamma > 0$,

$$\text{minimize} \quad f(x) + \gamma g(x)$$

- we desire both f and g to be small but they are weighed in by a given weight, γ (or often called penalty parameter)
- as γ is higher, we penalize more on g, then the minimized g is smaller; in this case, we care less about f
- appear in model performance evaluation where two diffferent metrics are desired to be small
- example 1: minimize model error + model complexity
- example 2: minimize system tracking error + input power

Multi-objective optimization

setting: minimizing $f_0 : \mathbf{R}^n \to \mathbf{R}^m$ (vector-valued function) over a feasible set

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

a vector optimization has a **vector-valued** objective function

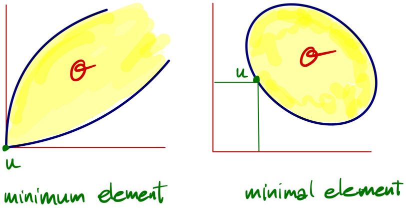
- example: $f_0(x) = (\text{fuel,time})$ the energy used and time spent of a vehicle parameter x
- require a generalized inequality definition for comparing any two vectors of $f_0(x)$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \preceq \begin{bmatrix} 10 \\ 3 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \not\preceq \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

here, for $f_0(x) \in \mathbf{R}^n$, we typically use the **non-negative orthant** to define $\preceq$

Achievable objective values

define $\mathcal{O} = \{ f_0(x) \mid x \in \mathcal{C} \}$ the set of objective values of feasible points



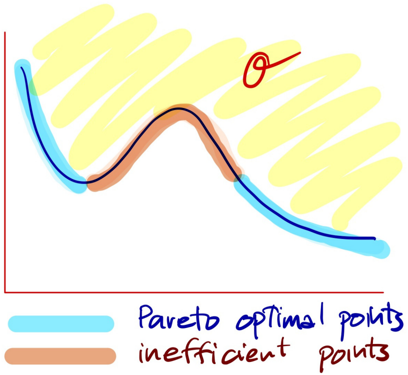
- u is said to be the **minimum** element of $\mathcal{O}$ if $u \preceq v$, for every $v \in \mathcal{O}$
- u is said to be a **minimal** element of $\mathcal{O}$ if $v \in \mathcal{O}$, $v \preceq u$ only if $v = u$
- if $\mathcal{O}$ has a minimum point (then it is unique) and

$$\exists \text{ feasible } x \quad \text{such that} \quad f_0(x) \preceq f_0(y), \quad \text{for all feasible } y$$

then we say x is **optimal**

Pareto optimal points

consider when $\mathcal{O}$ does not have a minimum element



- x is called **Pareto optimal** (or efficient) if $f_0(x)$ is a minimal element of $\mathcal{O}$
- a technique to extract pareto optimal points: scalarization (more on this later)

Optimality conditions

Unconstrained optimality

assumption: f is twice continuously differentiable (smooth objective)

- **necessary condition:** if $x^\star$ is a local minimizer of f then
  1. $\nabla f(x^\star) = 0$
  2. $\nabla^2 f(x^\star) \succeq 0$ (positive semidefinite)
- **sufficient condition:** if $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star) \succ 0$ (positive definite), then $x^\star$ is a strict local minimizer of f
- when f is convex and differentiable, any stationary point $x^\star$ is a global minimizer of f

example: the Rosenbrock function:

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

verify that $x^\star = (1, 1)$ is the only local minimizer of f

Constrained optimality

first, define the Lagrangian function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

where λ, ν are called the **Lagrange multipliers** for inequality and equality constraints

the KKT conditions are **necessary conditions** for optimality

1. zero-gradient condition of L: $\nabla_x L(x^\star, \lambda^\star, \nu^\star) = 0$
2. primal and dual feasibility

$$f_i(x^\star) \leq 0, i = 1, \ldots, m, \quad h_i(x^\star) = 0, i = 1, \ldots, p, \quad \lambda^\star \succeq 0$$

3. complementary slackness condition: $\lambda_i f_i(x) = 0$ for $i = 1, 2, \ldots, m$

fact: for convex problems, KKT conditions are sufficient and necessary for optimality

Optimality of contrained LS

derive KKT conditions for

$$\underset{x}{\text{minimize}} \quad (1/2)\|Ax - y\|_2^2 \quad \text{subject to} \quad l \preceq x \preceq u$$

the Lagrangian is $L(x, \lambda_1, \lambda_2) = (1/2)\|Ax - y\|_2^2 + \lambda_1^T(l - x) + \lambda_2^T x - u)$

KKT conditions are

1. zero-gradient of L: $A^T(Ax - y) - \lambda_1 + \lambda_2 = 0$
2. primal feasibility: $l \preceq x \preceq u$
3. dual feasibility: $\lambda_1, \lambda_2 \succeq 0$
4. complementary slackness condition:

$$\lambda_{1i}(l_i - x_i) = 0, \quad \lambda_{2i}(x_i - u_i) = 0, \quad i = 1, 2, \ldots, n$$

Intro to duality theory

some quick facts

- define the dual function as the infimum of the Lagrangian over primal variables

$$g(\lambda, \nu) = \inf_{x \in \mathbf{dom}\, \mathcal{D}} L(x, \lambda, \nu)$$

- for any $\lambda \succeq 0$, the dual function provides a lower bound for $p^\star$, *i.e.*, $g(\lambda, \nu) \leq p^\star$
- any optimization problem (called a primal problem) has its dual problem

$$\underset{\lambda, \nu}{\text{maximize}} \quad g(\lambda, \nu) \quad \text{subject to} \quad \lambda \succeq 0$$

  which is the problem of finding the *best* lower bound, denoted as $d^\star$, for $p^\star$
- more theoretical results about relations between primal and dual problems – when $d^\star = p^\star$, we say we have strong duality
- solving the dual can be more beneficial in some cases

Numerical methods

Overview of available methods

- unconstrained problems: gradient descent, Newton, quasi Newton, trust-region
- convex programs: interior point, gradient projection, ellipsoid method
- convex programs of certain structures: proximal methods
- linear programming: simplex, interior point
- quadratic programming: interior point, active set, conjugate gradient, augmented Lagrangian

Softwares

MATLAB: cvx

- CVX is a MATLAB-based modeling system for convex optimization
- `http://cvxr.com/cvx/`

Python

- **CVXPY**: Python-embedded modeling language for convex optimization problems available at `https://www.cvxpy.org/` by Stephen Boyd group
- **CVXOPT**: Python-based package for convex optimization available at `http://cvxopt.org/` by M. Andersen, J. Dahl and L. Vandenberghe

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