Feedback Stabilization of One-Link Flexible Robot Arms: An Infinite-Dimensional System Approach

J. Songsiri and W. Khovidhungij*
Department of Electrical Engineering,
Chulalongkorn University Bangkok 10330 Thailand

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Abstract

This research concerns the design of a controller for a flexible robot arm, which is modelled as a flexible beam clamped to a motor at the one end and free at the other end. A mass is also attached to the free end of the beam. This system can be described by an Euler-Bernoulli partial differential equation with initial and boundary conditions. In order to reduce the vibration of the tip mass, we apply a control law, which is a linear combination of the tip deflection and a linear functional of the beam deflection, through the motor acceleration. We show that the infinitesimal generator of the closed-loop system generates a contraction semigroup. Since the spectrum of a closed operator need not to have only the eigenvalues, it is rather difficult to analyze the stability of the system by using spectrum analysis approach. One of the techniques which is used here is to prove that the spectrum set consists only of eigenvalues by using Sobolev Imbedding theorem. We then prove that the closed-loop system is asymptotically stable.

Keywords: flexible beam, infinite-dimensional, semigroup

1 Introduction

In this paper, we consider a flexible robot arm, which is modelled as a flexible beam clamped to a motor at the one end and free at the other end. A mass is also attached to the free end of the beam. The behavior of the system can be described by an Euler-Bernoulli partial differential equation, together with appropriate initial and boundary conditions. Thus, the system is infinite-dimensional.

In general, we can formulate a linear infinite-dimensional control system into an abstract Cauchy problem on a Banach space (or Hilbert space) Z

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0 \in D(A) \tag{1}$$

where A is a closed operator with $D(A)$ dense in Z. The solution of this problem is

$$z(t) = T(t)z_0 + \int_0^t T(t-s)u(s)ds \tag{2}$$

where $T(t)$ is a C_0-*semigroup* of bounded operator on Z and it is the general form of e^{At} in case of finite-dimensional systems.

*Author to whom all correspondence should be addressed. Email: `advmath@hotmail.com`. Tel: +66-2 218-6487, Fax: +66-2 251-8991.

One way to design a controller for an infinite-dimensional control system is that we find a finite-dimensional model, and then design the controller for this approximated model. However, neglecting the high frequency dynamics by using an approximated model may lead to a "spillover" effect, which can destroy the stability of the original system. One of the papers describing about this effect is Bontsema and Curtain [1]. They showed that spillover can only occur if the approximation error exceeds the robustness margin of the controller. Moreover, Ballas [2] examined the spillover effect due to uncontrolled high frequency modes, which lead to closed-loop instabilities. From this viewpoint, the controller design for the original system using the infinite-dimensional system approach is an alternative way that will be considered here.

In previous works about flexible robot arms using infinite-dimensional models, there are many ways to prove the (asymptotic or exponential) stability of the system. For example, Guo [3, 4] used Riesz basis approach to prove exponential stability. They showed that there is a sequence of generalized eigenfunctions of operator A forms Riesz basis for the state-space and the spectral determined growth condition holds. Chen et. al. [5] and Morgül [6, 7] used the energy multiplier method. The principle is to find the summation of the energy and a multiplier function. The latter should be chosen in such a way that the summation, which represents the energy of state variables, decreases exponentially. The choices of the energy multiplier function depend on the system equations and the boundary conditions. Luo [8] proposed a direct strain feedback and introduced a concept of A-dependent operator, which accounts for the proof of the existence, uniqueness, and stability of the solution. The proof details were discussed later in [9]. Luo et. al. [9] employed the frequency domain approach. They proved that norm of the resolvent operator is uniformly bounded to obtain the exponential stability. Matsuno et. al. [10, 11] applied LaSalle's invariance principle which is the extension of Lyapunov stability to the infinite-dimensional systems.

However, in the above-mentioned works, the effects of the tip mass and the motion of the motor were not at once included in the mathematical model. Therefore, in this work, we will consider them simultaneously and propose a control law to stabilize the system. Here we apply a feedback through the angular acceleration of the motor to reduce the vibration of the tip mass. The proposed control law is a linear combination of the tip deflection and a linear functional of the beam deflection. We then prove that the closed-loop system is asymptotically stable. The remainder of this paper is organized as follows.

In section 2, we consider a flexible beam system. The equations of motion can be represented by partial differential equations with boundary conditions, which are examined in [9, 12, 13]. We then proposed the control law and formulate the closed-loop system equation into standard form. In section 3, we first investigate the properties of the infinitesimal generator. We prove that it generates a contraction semigroup and its spectrum consists of only isolated eigenvalues. Then the asymptotic stability proof is obtained from the spectrum analysis by showing that the real parts of the eigenvalues are less than zero. Finally, the conclusion is given in section 4.

2 System Equations

Consider a flexible beam in Fig 1, where $w(t)$ is the deflection of the beam and $\theta(t)$ is the motor angle. The equations of motion for this system are given by

$$\ddot{w}(x,t) + \frac{EI}{\rho} w''''(x,t) = -x\ddot{\theta}(t) \quad 0 < x < l\ ,\ t > 0, \tag{3}$$

$$w(0,t) = w'(0,t) = w''(l,t) = 0, \tag{4}$$

$$m\left[\ddot{w}(x,t) + l\ddot{\theta}(t)\right] = EIw'''(l,t), \tag{5}$$

$$I_{\mathrm{H}}\ddot{\theta}(t) = \tau(t) + EIw''(0,t). \tag{6}$$

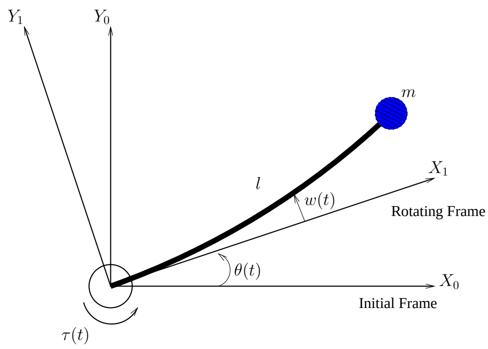

Figure 1: Flexible beam

where the constants EI, ρ, m, I_{H} and l are the physical parameters of the system.

We apply the feedback control law

$$\tau(t) = -EIw''(0,t) + KI_{\mathrm{H}}\left(\rho\left\langle \dot{w}, x\right\rangle + ml\dot{w}(l,t)\right) \tag{7}$$

where $K > 0$ is a constant. Substituting (7) into (6), we get the closed-loop system

$$\ddot{w}(x,t) + \frac{EI}{\rho}w''''(x,t) = -xK\left(\rho\left\langle \dot{w}, x\right\rangle + ml\dot{w}(l,t)\right), \tag{8}$$

$$w(0,t) = w'(0,t) = w''(l,t) = 0, \tag{9}$$

$$m\ddot{w}(x,t) + mlK\left(\rho\left\langle \dot{w}, x\right\rangle + ml\dot{w}(l,t)\right) = EIw'''(l,t). \tag{10}$$

Let us introduce a Hilbert space

$$H_0^2(0,l) = \left\{u \in H^2(0,l) \mid u(0) = u'(0) = 0\right\} \tag{11}$$

with a norm $\|u\|_{H_0^2} = \|u''\|_{L_2}$ and consider the Hilbert space $\mathcal{H} = H_0^2(0,l) \oplus L_2(0,l) \oplus \mathbb{C}$ with an inner product,

$$\langle u, v\rangle = EI\left\langle u_1'', v_1''\right\rangle_H + \rho\left\langle u_2, v_2\right\rangle_H + m\left\langle u_3, v_3\right\rangle_{\mathbb{C}}. \tag{12}$$

To prove that $\mathcal{H}$ is a Hilbert space, we will use lemma 2.1, theorem 2.2, lemma 2.3 and lemma 2.4 respectively. Firstly, the proof that $H_0^2(0,l)$ is a Hilbert space needs lemma 2.1 as follows.

Lemma 2.1 $H_0^2(0,l)$ is a closed subspace of $H^2(0,l)$.

Proof. Let $z_n \in H_0^2(0,l)$ and $z_n \to z$

$$\|z_n - z\|_2^2 + \|z_n' - z'\|_2^2 + \|z_n'' - z''\|_2^2 \to 0 \ \ , \ n \to \infty$$

Therefore, each term must converge to zero. Since $z_n \to z$ in L_2 norm, there is a subsequence z_{n_k} which converges to z almost everywhere. From the Sobolev Imbedding theorem, $H_0^2(0,l) \subset C_B^1(0,l)$. We can say that z_n is a continuous function and so z_{n_k} is. As a result, z_{n_k} converges to z everywhere. Consequently, we can conclude that

$$z_{n_k}(0) = 0 \Longrightarrow z(0) = 0$$

Similarly, we get

$$z_{n_k}'(0) = 0 \Longrightarrow z'(0) = 0$$

That is $z \in H_0^2(0,l)$. It shows that $H_0^2(0,l)$ is a closed subspace of $H^2(0,l)$ □

Since every close subspace of a complete space is also complete, $H_0^2(0,l)$ is a Hilbert space with the norm defined by the conventional Sobolev norm ($\|u\|_{H^2}^2 = \|u\|^2 + \|u'\|^2 + \|u''\|^2$). Next, we will show that a newly-defined norm,

$$\|u\|_{H_0^2}^2 = \|u''\|^2$$

is equivalent to the conventional one in lemma 2.4 by applying theorem 2.2 and lemma 2.3 as follows.

Theorem 2.2 Let Ω be an open interval in $\mathbb{R}$ and if $u \in H^1(\Omega)$, then u is absolutely continuous.
Proof. see [14]

Lemma 2.3

$$\|w\|^2 \leq l^4\|w''\|^2, \quad \forall w \in H_0^2(0,l) \tag{13}$$

Proof. From theorem 2.2 we can write the following,

$$w(x) = \int_0^x w'(x)dx + w(0).$$

Then,

$$\begin{aligned} |w(x)| &\leq \int_0^x |w'(x)|dx \leq \int_0^l |w'(x)|dx \\ |w(x)|^2 &\leq \left[\int_0^l |w'(x)|dx\right]^2 \leq \quad l\|w'\|^2 \end{aligned}$$

That is,

$$\|w\|^2 = \int_0^l |w(x)|^2dx \leq l^2\|w'\|^2 \tag{14}$$

Similarly,

$$\begin{aligned} |w'(x)| &\leq \int_0^x |w''(x)|dx \leq \int_0^l |w''(x)|dx \\ |w'(x)|^2 &\leq \left[\int_0^l |w''(x)|dx\right]^2 \leq l\|w''(x)\|^2 \\ \|w'\|^2 = \int_0^l |w'(x)|^2dx &\leq \quad l^2\|w''\|^2 \end{aligned} \tag{15}$$

The proof is complete by applying (15) to (14). □

Lemma 2.4 Define $\|w\|_{H_0^2}^2 = \|w''\|^2$, and we have the following.

$$\|\cdot\|_{H_0^2} \sim \|\cdot\|_{H^2}$$

Proof. If $w \in H_0^2(0,l)$, from lemma 2.3 we get,

$$\|w\|^2 + \|w'\|^2 + \|w''\|^2 \leq (l^4 + l^2 + 1)\|w''\|^2$$

and since

$$\|w''\|^2 \leq \|w\|^2 + \|w'\|^2 + \|w''\|^2,$$

we have

$$\frac{(\|w\|^2 + \|w'\|^2 + \|w''\|^2)}{(l^4 + l^2 + 1)} \leq \|w''\|^2 \leq \|w\|^2 + \|w'\|^2 + \|w''\|^2$$

It shows that $\|\cdot\|_{H_0^2} \sim \|\cdot\|_{H^2}$ □

Thus, from lemma 2.4, $H_0^2(0,l)$ is also a Hilbert space with the newly-defined norm. Therefore, the proof that $\mathcal{H}$ is a Hilbert space can be explained as follows.

Theorem 2.5 $\mathcal{H}$ is a Hilbert space.

Proof. Let $z_n = (z_{1n}, z_{2n}, z_{3n})$ be a Cauchy sequence in $\mathcal{H}$,

$$EI\|z''_{1n} - z''_{1m}\|^2 + \rho\|z_{2n} - z_{2m}\|^2 + m|z_{3n} - z_{3m}|^2 \to 0 \quad n, m \to \infty$$

Consider the first term in the left, since this norm is equivalent to the Sobolev norm and $H_0^2(0,l)$ is a Hilbert space, z_{1n} converges to a member in $H_0^2(0,l)$, says z_1. Likewise, since $L_2(0,l)$ and $\mathbb{C}$ are Hilbert spaces, z_{2n} converges to z_2 in $L_2(0,l)$ and z_{3n} converges to z_3 in $\mathbb{C}$. Hence, $\|z_n - z\|^2_{\mathcal{H}} \to 0$, where $z \in \mathcal{H}$. This means every Cauchy sequence in $\mathcal{H}$ converges to a member in $\mathcal{H}$. Thus, $\mathcal{H}$ is a Hilbert space. □

Subsequently, we can write (8)-(10) in the form $\dot{u} = \mathcal{A}u$ where,

$$u(t) = \begin{bmatrix} w(\cdot,t) & \dot{w}(\cdot,t) & \dot{w}(l,t) \end{bmatrix}^T \in \mathcal{H}$$

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\frac{EI}{\rho}\frac{\partial^4}{\partial x^4} & -Kx\rho\langle \cdot, x\rangle & -Kxml \\ \frac{EI}{m}\frac{\partial^3}{\partial x^3}\big|_{x=l} & -Kl\rho\langle\cdot, x\rangle & -Klml \end{bmatrix} \tag{16}$$

$$D(\mathcal{A}) = \left\{(u_1, u_2, u_3) \in H^4(0,l) \oplus H_0^2(0,l) \oplus \mathbb{C} \mid u_1(0) = u_1'(0) = u_1''(l) = 0, u_2(l) = u_3\right\} \tag{17}$$

Note that $\mathcal{A}$ is an unbounded operator on this Hilbert space $\mathcal{H}$. In the next section, we will show that $\mathcal{A}$ generates a C_0 semigroup.

3 Main Results

In this section, we will describe the main results of the paper as follows:

The first result is to show that $\mathcal{A}$ in (16) is an infinitesimal generator of a contraction semigroup by applying the following theorem.

Theorem 3.1 [15] Let A be a closed operator with $D(A)$ dense in Z. If

$$\mathrm{Re}\,\langle Az, z\rangle \le \omega\|z\|^2 \qquad \forall z \in D(A) \tag{18}$$

$$\mathrm{Re}\,\langle A^*z, z\rangle \le \omega\|z\|^2 \qquad \forall z \in D(A^*) \tag{19}$$

then A is the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \le e^{\omega t}$.

Lemma 3.2 Let $\mathcal{A}$ be defined as in (16). Then,

i. $\mathcal{A}^{-1}$ exists and is bound on $\mathcal{H}$.

ii. $\mathcal{A}$ is a densely defined closed operator in $\mathcal{H}$.

Proof. (i) A direct computation reveals that

$$\mathcal{A}^{-1}v = \begin{bmatrix} \frac{K}{EI}q_2(x)[\rho\langle v_1, x\rangle + mlv_1(l)] - \frac{\rho}{EI}\int_0^x\int_0^{x_4}\int_{x_3}^l\int_{x_2}^l v_2(x_1)dx_1dx_2dx_3dx_4 + \frac{m}{EI}q_1(x)v_3 \\ v_1(x) \\ v_1(l) \end{bmatrix} \tag{20}$$

where

$$q_1(x) = \frac{x^3}{6} - \frac{lx^2}{2}, \qquad q_2(x) = \rho\left(\frac{l^2x^3}{12} - \frac{l^3x^2}{6} - \frac{x^5}{120}\right) + mlq_1(x).$$

Let $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T = \mathcal{A}^{-1}v$. Consider its norm,

$$u_1''(x) = \frac{K}{EI}\{\rho\langle v_1, x\rangle + mlv_1(l)\}q_2''(x) + \frac{m}{EI}q_1''(x)v_3 - \frac{\rho}{EI}\int_x^l\int_{x_2}^l v_2(x_1)dx_1dx_2.$$

It follows that

$$
\begin{aligned}
\|u_1''\|^2 &\leq 4\left\|\frac{K}{EI}\rho\left\langle v_1,x\right\rangle q_2''\right\|^2+4\left\|\frac{K}{EI}mlv_1(l)q_2''\right\|^2+4\left\|\frac{mv_3}{EI}q_1''\right\|^2+4\left\|\frac{\rho}{EI}\int_x^l\int_{x_2}^l v_2(x_1)dx_1dx_2\right\|^2\\
&\leq 4\left|\frac{K\rho}{EI}\right|^2\|q_2''\|^2|\left\langle v_1,x\right\rangle|^2+4\left|\frac{Kml}{EI}\right|^2\|q_2''\|^2|v_1(l)|^2\\
&\quad +4\left|\frac{m}{EI}\right|^2\|q_1''\|^2|v_3|^2+4\left|\frac{\rho}{EI}\right|^2\int_0^l\left|\int_x^l\int_{x_2}^l v_2(x_1)dx_1dx_2\right|^2dx.
\end{aligned}
$$

Since $q_1'', q_2'' \in L_2(0,l)$, we can define

$$
\begin{aligned}
C_1 &= 4\left|\frac{K\rho}{EI}\right|^2\|q_2''\|^2, \qquad C_2 = 4\left|\frac{Kml}{EI}\right|^2\|q_2''\|^2,\\
C_3 &= 4\left|\frac{m}{EI}\right|^2\|q_1''\|^2, \qquad C_4 = 4\left|\frac{\rho}{EI}\right|^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|u_1''(x)\|^2 &\leq C_1\|v_1\|^2\cdot\frac{l^3}{3}+C_2|v_1(l)|^2+C_3|v_3|^2+C_4l\sup_{x\in(0,l)}\left|\int_x^l\int_{x_2}^l v_2(x_1)dx_1dx_2\right|^2\\
&\leq \frac{C_1l^3}{3}l^4\|v_1''\|^2+C_2'\|v_1''\|^2+C_3|v_3|^2+C_4l\sup_{x\in(0,l)}\int_x^l\left|\int_{x_2}^l v_2(x_1)dx_1\right|^2dx_2 \qquad (21)\\
&\leq \frac{C_1l^7}{3}\|v_1''\|^2+C_2'\|v_1''\|^2+C_3|v_3|^2+C_4l\int_0^l\left|\int_{x_2}^l v_2(x_1)dx_1\right|^2dx_2\\
&\leq \left(\frac{C_1l^7}{3}+C_2'\right)\|v_1''\|^2+C_3|v_3|^2+C_4l^2\sup_{x_2\in(0,l)}\left|\int_{x_2}^l v_2(x_1)dx_1\right|^2\\
&\leq \left(\frac{C_1l^7}{3}+C_2'\right)\|v_1''\|^2+C_3|v_3|^2+C_4l^2\sup_{x_2\in(0,l)}\int_{x_2}^l|v_2(x_1)|^2dx_1\\
&\leq \left(\frac{C_1l^7}{3}+C_2'\right)\|v_1''\|^2+C_3|v_3|^2+C_4l^2\|v_2\|^2, \qquad (22)
\end{aligned}
$$

where the first and second term in (21) comes from (13) and the Sobolev Imbedding theorem in (54) respectively.

For $u_2(x) = v_1(x)$ and using (13), we get

$$\|u_2\|^2 = \|v_1\|^2 \leq l^4\|v_1''\|^2. \qquad (23)$$

Similarly, for $u_3 = v_1(l)$ and from (54), we obtain

$$|u_3|^2 = |v_1(l)|^2 \leq C_5\|v_1''\|^2. \qquad (24)$$

From (22)-(24),

$$
\begin{aligned}
\|u\|_{\mathcal{H}}^2 &= EI\|u_1''\|^2+\rho\|u_2\|^2+m|u_3|^2\\
&\leq \left\{EI\left(\frac{C_1l^7}{3}+C_2'\right)+\rho l^4+mC_5\right\}\|v_1''\|^2+EIC_4l^2\|v_2\|^2+EIC_3|v_3|^2.
\end{aligned}
$$

There is always $M > 0$ such that $\|\mathcal{A}^{-1}v\|_{\mathcal{H}}^2 \leq M\|v\|_{\mathcal{H}}^2$. Thus, $\mathcal{A}^{-1}$ is a bounded operator.

(ii) As a result, from the Closed Graph theorem, $\mathcal{A}^{-1}$ is closed and so is $\mathcal{A}$. □

Lemma 3.3 The operator $\mathcal{A}$ defined in (16) generates a contraction semigroup.
Proof. By the definition of the adjoint operator, we have

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I & 0 \\ \frac{EI}{\rho}\frac{\partial^4}{\partial x^4} & -Kx\rho\langle\cdot, x\rangle & -Kxml \\ -\frac{EI}{m}\frac{\partial^3}{\partial x^3}|_{x=l} & -K\rho l\langle\cdot, x\rangle & -Klml \end{bmatrix}, \tag{25}$$

$$D(\mathcal{A}^*) \quad = \quad \left\{(v_1, v_2, v_3) \in H^4(0, l) \oplus H_0^2(0, l) \oplus \mathbb{C} \mid v_2(0) = v_2'(0) = v_1''(l) = 0, v_3 = v_2(l)\right\}.$$

Consider

$$\begin{aligned} \langle \mathcal{A}u, u\rangle_{\mathcal{H}} &= EI\langle u_2'', u_1''\rangle + \rho\left\langle -\frac{EI}{\rho}u_1'''', u_2\right\rangle - \rho\langle Kx[\rho\langle u_2, x\rangle + mlu_3], u_2\rangle \\ &\quad + m\left\langle -Kl[\rho\langle u_2, x\rangle + mlu_3] + \frac{EI}{m}u_1'''(l), u_3\right\rangle_{\mathbb{C}} \\ &= EI\overline{\langle u_1'', u_2''\rangle} - EI\langle u_1'', u_2''\rangle - EIu_1'''(l)\overline{u_2(l)} + EIu_1'''(l)\overline{u_3} \\ &\quad - K[\rho\langle u_2, x\rangle + mlu_3]\left(\rho\overline{\langle u_2, x\rangle} + ml\overline{u_3}\right) \\ &= EI\overline{\langle u_1'', u_2''\rangle} - EI\langle u_1'', u_2''\rangle - K|\rho\langle u_2, x\rangle + mlu_3|^2 \end{aligned}$$

Therefore,

$$\text{Re}\langle \mathcal{A}u, u\rangle_{\mathcal{H}} = -K|\rho\langle u_2, x\rangle + mlu_3|^2 \leq 0. \tag{26}$$

Similarly, from the adjoint operator of $\mathcal{A}$ in (25)

$$\begin{aligned} \langle \mathcal{A}^*u, u\rangle_{\mathcal{H}} &= -EI\langle u_2'', u_1''\rangle + \rho\left\langle \frac{EI}{\rho}u_1'''', u_2\right\rangle - \rho\langle Kx[\rho\langle u_2, x\rangle + mlu_3], u_2\rangle \\ &\quad + m\left\langle -Kl[\rho\langle u_2, x\rangle + mlu_3] - \frac{EI}{m}u_1'''(l), u_3\right\rangle_{\mathbb{C}} \\ &= -EI\overline{\langle u_1'', u_2''\rangle} + EI\langle u_1'', u_2''\rangle + EIu_1'''(l)u_2(l) - EIu_1'''(l)\overline{u_3} \\ &\quad - K[\rho\langle u_2, x\rangle + mlu_3]\left[\rho\overline{\langle u_2, x\rangle} + ml\overline{u_3}\right] \\ &= -EI\overline{\langle u_1'', u_2''\rangle} + EI\langle u_1'', u_2''\rangle - K|\rho\langle u_2, x\rangle + mlu_3|^2 \end{aligned}$$

Thus,

$$\text{Re}\langle \mathcal{A}^*u, u\rangle_{\mathcal{H}} = -K|\rho\langle u_2, x\rangle + mlu_3|^2 \leq 0. \tag{27}$$

Since $\mathcal{A}$ is closed with $D(\mathcal{A})$ dense in $\mathcal{H}$ and from (26)-(27), (18)-(19) is satisfied with $\omega = 0$. This shows that A generates the contraction semigroup, $\|T(t)\| \leq 1$. □

Next, we will show that the spectrum of $\mathcal{A}$, indeed, consists of only isolated eigenvalues with finite multiplicity by applying the following theorem.

Theorem 3.4 [15] Let A be a closed linear operator with $0 \in \rho(A)$ and A^{-1} is compact. The spectrum of A consists of only isolated eigenvalues with finite multiplicity.

Lemma 3.5 $\mathcal{A}^{-1}$ is compact.
Proof. The expression of $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ in (20) can be written in the following form

$$\mathcal{A}^{-1} = \begin{bmatrix} T_1 & T_2 & T_3 \\ I & 0 & 0 \\ T_4 & 0 & 0 \end{bmatrix}$$

If all T_i's are compact operators, then $\mathcal{A}^{-1}$ is compact. We will prove the compactness property of each T_i as follows:
1. Consider $T_1 : H_0^2(0,l) \to H_0^2(0,l)$ defined by

$$T_1 v = \frac{K}{EI} q_2(x)(\rho \langle v, x \rangle + mlv(l))$$

Let S_N be a bounded set of $v \in H_0^2(0,l)$ with $\|v\|_{H_0^2} \leq N$. Then,

$$\begin{aligned}
\|T_1 v\|_{H_0^2} &= \frac{K}{EI} \|q_2(x)\|_{H_0^2} |\rho \langle v, x \rangle + mlv(l)| \\
&\leq \frac{K}{EI} \|q_2(x)\|_{H_0^2} (\rho |\langle v, x \rangle| + ml|v(l)|) \\
&\leq \frac{K}{EI} \|q_2(x)\|_{H_0^2} \left\{ \rho l \sqrt{\frac{l}{3}} \|v\|_{L_2} + mlM_1 \|v\|_{H_0^2} \right\} \qquad (28) \\
&\leq \frac{K}{EI} \|q_2(x)\|_{H_0^2} \left\{ \rho l \sqrt{\frac{l}{3}} N' + mlM_1 N \right\} \qquad (29) \\
&\leq M_2
\end{aligned}$$

where (28) is obtained by using the Sobolev imbedding theorem (54) and the Cauchy-Schwarzt inequality. From the fact that $\|\cdot\|_{H^2} \sim \|\cdot\|_{H_0^2}$, we get (29). This shows that $T_1 v$ is uniformly bounded.

Since $q_2(x)$ is continuous, i.e., for all $x_0 \in (0,l)$ and $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \Rightarrow \|q_2(x) - q_2(x_0)\| < \epsilon_1,$$

we have

$$\begin{aligned}
\|T_1 v(x) - T_1 v(x_0)\| &= \frac{K}{EI} |\rho \langle v, x \rangle + mlv(l)| \|q_2(x) - q_2(x_0)\| \\
&\leq \frac{K}{EI} \left\{ \rho l \sqrt{\frac{l}{3}} N' + mlM_1 N \right\} \|q_2(x) - q_2(x_0)\|.
\end{aligned}$$

Let $\epsilon = EI\epsilon_1 / K(\rho l \sqrt{\frac{l}{3}} N' + mlM_1 N)$, so

$$|x - x_0| < \delta_1 \Rightarrow \|T_1 v(x) - T_1 v(x_0)\| < \epsilon$$

Notice that δ_1 does not depend on the choice of $v \in S_N$, which implies that $T_1 v$ is equicontinuous. From Arzela's theorem, the image of $T_1 v$ is a precompact set. Therefore, T_1 is compact.
2. Consider $T_2 : L_2(0,l) \to H_0^2(0,l)$ defined by

$$T_2 v = -\frac{\rho}{EI} \int_0^x \int_0^{x_4} \int_{x_3}^l \int_{x_2}^l v(x_1) dx_1 dx_2 dx_3 dx_4$$

Let $f \in L_2(0,l)$ and let χ_S be the characteristic function of a set S. We know that $\chi_{(0,x)} \in L_2[0,l] \times L_2[0,l]$. Thus, the operator A defined by

$$Af = \int_0^x f(\tau) d\tau = \int_0^l \chi_{(0,x)} f(\tau) d\tau$$

is a compact operator from $L_2(0,l) \to L_2(0,l)$ and T_2 can be considered as the composition of the operator A defined above. Since the compositions of compact operator are compact, we

can conclude that T_2 is compact.
3. Consider $T_3 : \mathbb{C} \to H_0^2(0,l)$ defined by

$$T_3 v = \frac{m}{EI} q_1(x) v$$

As in the case of T_1, we can see that T_3 is compact.
4. $I : H_0^2(0,l) \to L_2(0,l)$ is a compact operator. This can be proved by the Hilbert-Schmidt Imbedding Theorem (see Appendix).
5. $T_5 : H_0^2(0,l) \to \mathbb{C}$, $T_5 v = v(l)$
From the Sobolev Imbedding theorem [16], T_5 is a bounded linear functional. Its image has finite dimensional range, so T_5 is compact.

According to all of the above, we can conclude that $\mathcal{A}^{-1}$ is compact. □

Lemma 3.6 The spectrum of $\mathcal{A}$ in (16) consists of only isolated eigenvalues with finite multiplicity.
Proof. By the definition of the resolvent set, $0 \in \rho(\mathcal{A})$. The proof is completed following Lemma 3.5 and Theorem 3.4. □

In what follows, we analyze the eigenvalues of $\mathcal{A}$ by showing that all these eigenvalues lie on the open-left half complex plane. We first begin with the following lemma.

Lemma 3.7 If λ and $\phi(x) = \begin{bmatrix}\phi_1(x) & \phi_2(x) & \phi_3\end{bmatrix}^T$ are an eigenvalue and the corresponding eigenvector of $\mathcal{A}$ respectively, then

$$F(\phi_1) \equiv \rho \langle \phi_1, x \rangle + ml\phi_1(l) \neq 0.$$

Proof. The eigenvalue problem is to find nontrivial $\phi \in D(\mathcal{A})$ and $\lambda \in \mathbb{C}$ such that

$$\mathcal{A}\phi(x) = \lambda\phi(x).$$

From (16)-(17), we get the ordinary differential equation of $\phi_1(x)$ and the boundary conditions.

$$\phi_1''''(x) + \frac{\rho\lambda^2}{EI}\phi_1(x) = -\frac{\rho K \lambda}{EI}\left[\rho\langle\phi_1, x\rangle + ml\phi_1(l)\right]x \tag{30}$$

$$\phi_1(0) = \phi_1'(0) = \phi_1''(l) = 0 \tag{31}$$

$$\phi_1'''(l) = \frac{Kml\lambda}{EI}\left[\rho\langle\phi_1, x\rangle + ml\phi_1(l)\right] + \frac{m\lambda^2}{EI}\phi_1(l). \tag{32}$$

Next, we will find the solution $\phi_1(x)$ of equations (30)-(32) by assuming $\phi_1(x) = \phi_h(x) + \phi_p(x)$, where $\phi_h(x)$ and $\phi_p(x)$ satisfy

$$\phi_h''''(x) + \frac{\rho\lambda^2}{EI}\phi_h(x) = 0$$

$$\phi_p''''(x) + \frac{\rho\lambda^2}{EI}\phi_p(x) = -\frac{\rho K}{EI}\lambda\left[\rho\langle\phi_1, x\rangle + ml\phi_1(l)\right]\cdot x.$$

Let $\beta = (\rho/EI)^{1/4}\sqrt{\lambda i}$, we can solve $\phi_h(x)$ and $\phi_p(x)$ as follows.

$$\phi_h(x) = c_1\cosh(\beta x) + c_2\cos(\beta x) + c_3\sinh(\beta x) + c_4\sin(\beta x).$$

$$\phi_p(x) = -\frac{KF(\phi_1)}{\lambda}\cdot x.$$

Since $\phi_1(0) = \phi_h(x) + \phi_p(0) = 0$, then $c_1 + c_2 = 0$ and $\phi_h(x)$ becomes

$$\phi_h(x) = c_1[\cosh(\beta x) - \cos(\beta x)] + c_3\sinh(\beta x) + c_4\sin(\beta x). \tag{33}$$

Next, to be concise, we will use the following symbols.

$$\mathrm{s} \equiv \sin(\beta l) \quad \mathrm{c} \equiv \cos(\beta l) \quad \mathrm{sh} \equiv \sinh(\beta l) \quad \mathrm{ch} \equiv \cosh(\beta l)$$

Next, $\phi_1'(0) = \phi_h'(0) + \phi_p'(0) = 0$, and so

$$\beta(c_3 + c_4) - \frac{K}{\lambda} F(\phi_1) = 0. \tag{34}$$

Similarly, from $\phi_1''(l) = \phi_h''(l) + \phi_p''(l) = 0$, we get

$$\beta^2 \left\{ c_1(\mathrm{ch} + \mathrm{c}) + c_3 \cdot \mathrm{sh} - c_4 \cdot \mathrm{s} \right\} = 0. \tag{35}$$

From the last boundary condition in (32), we have

$$\begin{aligned} \beta^3 \left\{ c_1(\mathrm{sh} - \mathrm{s}) + c_3 \cdot \mathrm{ch} - c_4 \cdot \mathrm{c} \right\} &= \frac{Kml\lambda}{EI} F(\phi_1) + \frac{m\lambda^2}{EI} \left\{ c_1(\mathrm{ch} - \mathrm{c}) + c_3 \cdot \mathrm{sh} + c_4 \cdot \mathrm{s} - \frac{Kl}{\lambda} F(\phi_1) \right\} \\ &= \frac{m\lambda^2}{EI} \left\{ c_1(\mathrm{ch} - \mathrm{c}) + c_3 \cdot \mathrm{sh} + c_4 \cdot \mathrm{s} \right\} \end{aligned}$$

or

$$c_1 \left\{ \beta^3(\mathrm{sh} - \mathrm{s}) - \frac{m\lambda^2}{EI}(\mathrm{ch} - \mathrm{c}) \right\} + c_3 \left\{ \beta^3 \cdot \mathrm{ch} - \frac{m\lambda^2}{EI} \mathrm{sh} \right\} + c_4 \left\{ -\beta^3 \cdot \mathrm{c} - \frac{m\lambda^2}{EI} \mathrm{s} \right\} = 0$$

$$c_1 \left\{ \beta^3(\mathrm{sh} - \mathrm{s}) + \frac{m\beta^4}{\rho}(\mathrm{ch} - \mathrm{c}) \right\} + c_3 \left\{ \beta^3 \cdot \mathrm{ch} + \frac{m\beta^4}{\rho} \mathrm{sh} \right\} + c_4 \left\{ -\beta^3 \cdot \mathrm{c} + \frac{m\beta^4}{\rho} \mathrm{s} \right\} = 0. \tag{36}$$

Since $\lambda = 0 \notin \sigma_p(\mathcal{A})$, i.e., $\beta \neq 0$, (35)-(36) become

$$c_1(\mathrm{ch} + \mathrm{c}) + c_3 \cdot \mathrm{sh} - c_4 \cdot \mathrm{s} = 0 \tag{37}$$

$$c_1 \left\{ (\mathrm{sh} - \mathrm{s}) + \frac{m\beta}{\rho}(\mathrm{ch} - \mathrm{c}) \right\} + c_3 \left\{ \mathrm{ch} + \frac{m\beta}{\rho} \mathrm{sh} \right\} + c_4 \left\{ -\mathrm{c} + \frac{m\beta}{\rho} \mathrm{s} \right\} = 0. \tag{38}$$

The expression in (34) shows that we have to calculate $F(\phi_1)$ in term of c_1, c_2, c_3.

From the notation,

$$\begin{aligned} F(\phi_1) &= \rho \langle \phi_1, x \rangle + ml\phi_1(l) \\ &= \frac{c_1}{\beta^2}[\rho\beta l(\mathrm{sh} - \mathrm{s}) - \rho(\mathrm{ch} + \mathrm{c}) + 2\rho] + \frac{c_3}{\beta^2}[\rho\beta l \cdot \mathrm{ch} - \rho \cdot \mathrm{sh}] + \frac{c_4}{\beta^2}[-\rho\beta l \cdot \mathrm{c} + \rho \cdot \mathrm{s}] \\ &\quad - \frac{\rho K l^3}{3\lambda} F(\phi_1) + c_1 ml(\mathrm{ch} - \mathrm{c}) + c_3 ml \cdot \mathrm{sh} + c_4 ml \cdot \mathrm{s} - \frac{Kml^2}{\lambda} F(\phi_1) \end{aligned}$$

or

$$\begin{aligned} F(\phi_1) &\left(1 + \frac{\rho K l^3}{3\lambda} + \frac{Kml^2}{\lambda} \right) \\ &= \frac{c_1}{\beta^2}[\rho\beta l(\mathrm{sh} - \mathrm{s}) - \rho(\mathrm{ch} + \mathrm{c}) + 2\rho + ml\beta^2(\mathrm{ch} - \mathrm{c})] \\ &\quad + \frac{c_3}{\beta^2}[\rho\beta l \cdot \mathrm{ch} - \rho \cdot \mathrm{sh} + ml\beta^2 \cdot \mathrm{sh}] + \frac{c_4}{\beta^2}[-\rho\beta l \cdot \mathrm{c} + \rho \cdot \mathrm{s} + ml\beta^2 \cdot \mathrm{s}]. \end{aligned}$$

Therefore,

$$F(\phi_1) = \frac{\lambda[f_1(\lambda)c_1 + f_3(\lambda)c_3 + f_4(\lambda)c_4]}{g(\lambda)} \tag{39}$$

where

$$\begin{aligned} f_1(\lambda) &= \rho\beta l(\mathrm{sh}-\mathrm{s}) - \rho(\mathrm{ch}+\mathrm{c}) + 2\rho + ml\beta^2(\mathrm{ch}-\mathrm{c}) \\ f_3(\lambda) &= \rho\beta l\cdot\mathrm{ch} - \rho\cdot\mathrm{sh} + ml\beta^2\cdot\mathrm{sh} \\ f_4(\lambda) &= -\rho\beta l\cdot\mathrm{c} + \rho\cdot\mathrm{s} + ml\beta^2\cdot\mathrm{s} \\ g(\lambda) &= \beta^2(\lambda + \frac{\rho K l^3}{3} + Kml^2). \end{aligned}$$

Sincer $F(\phi)$ is calculated, we are now ready to prove by constradiction. Let $F(\phi_1) \equiv \rho\langle\phi_1, x\rangle + ml\phi_1(l) = 0$. From (34), we get $c_4 = -c_3$. Substitute it into (39),(37) and (38), we have

$$c_1(\mathrm{ch}+\mathrm{c}) + c_3(\mathrm{sh}+\mathrm{s}) = 0 \tag{40}$$

$$c_1\left\{(\mathrm{sh}-\mathrm{s}) + \frac{m\beta}{\rho}(\mathrm{ch}-\mathrm{c})\right\} + c_3\left\{(\mathrm{ch}+\mathrm{c}) + \frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s})\right\} = 0 \tag{41}$$

$$\begin{aligned} &c_1\left\{\rho l\beta(\mathrm{sh}-\mathrm{s}) - \rho(\mathrm{ch}+\mathrm{c}) + 2\rho + ml\beta^2(\mathrm{ch}-\mathrm{c})\right\} \\ &\qquad + c_3\left\{\rho l\beta(\mathrm{ch}+\mathrm{c}) - \rho(\mathrm{sh}+\mathrm{s}) + ml\beta^2(\mathrm{sh}-\mathrm{s})\right\} = 0. \end{aligned} \tag{42}$$

We will show that (40)-(41) have only one solution, that is, $c_1 = c_3 = 0$, by arranging them as follows.

$$\begin{bmatrix} (\mathrm{ch}+\mathrm{c}) & (\mathrm{sh}+\mathrm{s}) \\ (\mathrm{sh}-\mathrm{s}) + \frac{m\beta}{\rho}(\mathrm{ch}-\mathrm{c}) & (\mathrm{ch}+\mathrm{c}) + \frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s}) \\ \rho l\beta(\mathrm{sh}-\mathrm{s}) - \rho(\mathrm{ch}+\mathrm{c}) + 2\rho + ml\beta^2(\mathrm{ch}-\mathrm{c}) & \rho l\beta(\mathrm{ch}+\mathrm{c}) - \rho(\mathrm{sh}+\mathrm{s}) + ml\beta^2(\mathrm{sh}-\mathrm{s}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is equivalent to

$$\begin{bmatrix} (\mathrm{ch}+\mathrm{c}) & (\mathrm{sh}+\mathrm{s}) \\ (\mathrm{sh}-\mathrm{s}) + \frac{m\beta}{\rho}(\mathrm{ch}-\mathrm{c}) & (\mathrm{ch}+\mathrm{c}) + \frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s}) \\ 2\rho & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $c_1 = 0$. It can be proved that the term $(\mathrm{sh}+\mathrm{s})$ and $(\mathrm{ch}+\mathrm{c}) + \frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s})$ are not simultaneously equal to zero at the same β. Therefore, $c_3 = 0$ and then $\phi_1(x) = 0$. As a result, $\phi(x)$ is not the eigenvector of $\mathcal{A}$, which is the contradiction. □

This result gives an important role in the proof that the real part of all eigenvalues are less than zero, which can be shown in the following lemma.

Lemma 3.8 For any $0 < K < \infty$, we have

$$\mathrm{Re}\,\lambda(\mathcal{A}) < 0.$$

Proof. According to the eigenvalue problem derived in (**??**)-(32), take the inner product with ϕ_1 on both sides in (30).

$$\langle\phi_1'''', \phi_1\rangle + \frac{\rho\lambda^2}{EI}\langle\phi_1, \phi_1\rangle + \frac{\rho K\lambda}{EI}(\rho\langle\phi_1, x\rangle + ml\phi_1(l))\langle x, \phi_1\rangle = 0. \tag{43}$$

Since

$$\begin{aligned} \langle\phi_1'''', \phi_1\rangle &= \int_0^l \phi_1''''\overline{\phi_1}dx \\ &= \phi_1'''\overline{\phi_1}\,|_0^l - \int_0^l \phi_1'''\overline{\phi_1'}dx \\ &= \phi_1'''(l)\overline{\phi_1(l)} - \phi_1''\overline{\phi_1'}\,|_0^l + \int_0^l \phi_1''\overline{\phi_1''}dx \\ &= \phi_1'''(l)\overline{\phi_1(l)} + \|\phi''\|^2 \\ &= \lambda\frac{\rho Kml}{EI}\langle\phi_1, x\rangle\overline{\phi_1(l)} + \lambda\frac{Km^2l^2}{EI}|\phi_1(l)|^2 + \lambda^2\frac{m}{EI}|\phi_1(l)|^2 + \|\phi''\|^2, \end{aligned} \tag{44}$$

and by substituting (44) in (43), we obtain

$$\lambda\rho Kml\langle\phi_1,x\rangle\overline{\phi_1(l)}+\lambda Km^2l^2|\phi_1(l)|^2+\lambda^2 m|\phi_1(l)|^2+EI\|\phi''\|^2+\rho\lambda^2\|\phi_1\|^2 \\ +\lambda\rho^2K|\langle\phi_1,x\rangle|^2+\lambda\rho Kml\phi_1(l)\langle x,\phi_1\rangle=0.$$

$$\lambda^2\left\{m|\phi_1(l)|^2+\rho\|\phi_1\|^2\right\}+EI\|\phi''\|^2+\lambda K\left\{\rho^2|\langle\phi_1,x\rangle|^2+m^2l^2|\phi_1(l)|^2\right\} \\ +\lambda K\left\{2\rho ml\mathrm{Re}(\phi_1(l)\langle x,\phi_1\rangle)\right\}=0.$$

$$\lambda^2\left\{m|\phi_1(l)|^2+\rho\|\phi_1\|^2\right\}+\lambda K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2+EI\|\phi''\|^2=0. \tag{45}$$

Let $\lambda=a+ib$, then (45) can be written as

$$(a^2-b^2+i2ab)\left\{m|\phi_1(l)|^2+\rho\|\phi_1\|^2\right\}+EI\|\phi''\|^2+(a+ib)K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2=0, \tag{46}$$

which can be splitted into two equations as

$$(a^2-b^2)(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+EI\|\phi''\|^2+a\cdot K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2=0, \tag{47}$$

and

$$2ab(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+b\cdot K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2=0. \tag{48}$$

First, suppose $b=0$ in (47), we have

$$a^2(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+a\cdot K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2+EI\|\phi''\|^2=0. \tag{49}$$

According to lemma 3.7, $|\rho\langle\phi_1,x\rangle+ml\phi_1(l)|$ is not equal to zero. Then all coefficients of a are real number greater than zero. Therefore, all roots a of (49) are less than zero.

Next, when $b\neq 0$ in (48),

$$a=-\frac{K\left|\rho\langle\phi_1,x\rangle+ml\phi_1(l)\right|^2}{2(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)}<0 \tag{50}$$

Therefore, $\mathrm{Re}(\lambda)<0$. □

Finally, to prove the closed-loop stability we need the following theorem to show the asymptotic stability of the semigroup generated by $\mathcal{A}$.

Theorem 3.9 [17, 18] Let $T(t)$ be a uniformly bounded semigroup on a Banach space X with an infinitesimal generator A. Suppose that

i. $\sigma(A)\cap i\mathbb{R}$ is countable,

ii. $\sigma_P(A^*)\cap i\mathbb{R}=\emptyset$,

then $T(t)$ is asymptotically stable.

Theorem 3.10 The semigroup generated by $\mathcal{A}$ is asymptotically stable.
Proof Since $\sigma(\mathcal{A})=\sigma_P(\mathcal{A})$ and when $K<\infty$, all eigenvalues lie on the open left-half plane, we can conclude that there is no eigenvalue on the imaginary axis. Thus, $\sigma(\mathcal{A})\cup i\mathbb{R}$ is an empty set, which is countable. Moreover,$\sigma_P(\mathcal{A}^*)=\sigma_r(\mathcal{A})=\emptyset$. The semigroup of closed-loop system is a contraction semigroup which is uniformly bounded. Therefore, the proof follows from Lemmas 3.3, 3.6, 3.8 and Theorem 3.9.

4 Conclusion

In this work, we consider the design of a controller for a one-link flexible robot arm, modelled as an infinite-dimensional system, where the effects of the tip mass and the motion of the motor were included simultaneously in the mathematical model. The proposed control law is a linear combination of the tip deflection and a linear functional of the beam deflection. It was shown that (1) the infinitesimal generator of the closed-loop system generates a contraction semigroup, (2) the spectrum of the generator consists of only isolated eigenvalues with finite multiplicity, and (3) the real parts of these eigenvalues are less than zero. Therefore, the closed-loop system is asymptotically stable.

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A Appendix

In this section, we mention the Sobolev Imbedding theorem [16] which is necessary and frequently used in this work.

Definition A.1 Let X and Y be a Banach space, X *is imbedded* in Y and use the notation $X \to Y$, if

i. X is a vector subspace of Y

ii. The identity operator $I : X \to Y$ is continuous, or equivalently, there is $M > 0$ such that

$$\|Ix\|_Y \leq M\|x\|_X, \quad \forall x \in X$$

Theorem A.2 (The Sobolev imbedding theorem) Let Ω be a domain in $\mathbb{R}$, $j, m \in \mathbb{N} \cup \{0\}$ and $1 \leq p < \infty$.

$$H^{j+m}(\Omega) \quad \to \quad C_B^j(\Omega) \tag{51}$$

$$H^{j+m}(\Omega) \quad \to \quad C^{j,\lambda}(\overline{\Omega}) \quad ,0 < \lambda \leq m - \frac{1}{2} \tag{52}$$

Theorem A.3 (The Hilbert-Schmidt Imbedding theorem) Let Ω be a domain in $\mathbb{R}$ and $m, k \in \mathbb{N} \cup \{0\}$ with $k > 1/2$, then the mapping

$$I : H^{m+k}(\Omega) \to H^m(\Omega) \tag{53}$$

is a compact operator.

Remarks

i. From (51), when $j = 3, m = 1$, $H^4(0, l) \to C_B^3(0, l)$. If $j = 1, m = 1$, then $H^2(0, l) \to C_B^1(0, l)$. In other words, a function in H^4 or H^2 can be considered continuous. Note that, the higher order Sobolev space is, the more differentiable a function in the space is.

ii. Consider (52), if $j = 0, m = 2, 0 < \lambda \leq 3/2$ and $\Omega = (0, l)$, then

$$H^2(0, l) \to C^{0,\lambda}[0, l]$$

From the definition of imbedding,

$$\|u\|_{C^{0,\lambda}[0,l]} \leq M\|u\|_{H^2(0,l)} \quad \forall u \in H^2(0,l)$$

where the norm in $C^{m,\lambda}(\overline{\Omega})$ is defined by

$$\|u; C^{m,\lambda}(\overline{\Omega})\| = \|u; C^m(\overline{\Omega})\| + \max_{0\leq|\alpha|\leq m} \sup_{x,y\in\Omega \;\; x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}$$

Therefore,

$$\begin{aligned}
\|u\|_{C[0,l]} &\leq M\|u\|_{H^2(0,l)} \\
\|u\|_{C[0,l]} &\leq M_1\|u\|_{H_0^2(0,l)} \quad \text{(because} \quad \|\cdot\|_{H_0^2} \sim \|\cdot\|_{H^2}) \\
\sup_{x\in[0,l]} |u(x)| &\leq M_1\|u''\| \\
|u(l)| &\leq M_1\|u''\| \;\; \forall u \in H_0^2(0,l) \qquad (54)
\end{aligned}$$

This shows that we can find the magnitude bound of a function in $H_0^2(0,l)$ limited by the norm in $H_0^2(0,l)$.

iii. From (53), if $m = 0, k = 2$, then $I : H^2(0,l) \to L_2(0,l)$ is a compact operator. Since $H_0^2(0,l) \subset H^2(0,l)$, $I : H_0^2(0,l) \to L_2(0,l)$ is also a compact operator.

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