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Feedback Stabilization of One-Link Flexible Robot Arms : An Infinite Dimensional System Approach.

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Using pdfslide package and P^4

Outline

- ➾ Introduction.
- ➾ Euler-Bernoulli beam equation.
- ➾ Infinite-Dimensional System Theory.
- ➾ Notation
- ➾ Sobolev Imbedding Theorem.
- ➾ The Closed-Loop System.

Introduction

Recent Research

- Model : 1. Tip mass 2. Motor angle
- Control Law : velocity or its spatial higher derivative feedback.
- Stability Analysis : Spectral growth-determined condition, Energy Multiplier Method, Frequency domain condition.

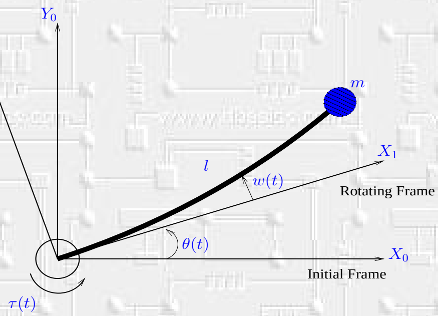
Work Procedure

- Study the theory of infinite dimensional control systems.
- Find a mathematical model of the flexible robot arm system.
- Propose a feedback control law.
- Analyze the closed-loop stability.
- Conclude the results.

The Benefit of this work

- To understand the properties of the flexible robot arm system.
- To propose a control law that guarantees the closed-loop stability of the system.

Mathematic model of Flexible beam



$$\ddot{w}(x,t) + EIw''''(x,t) + x\ddot{\theta}(t) = 0 \tag{1}$$

$$\tau + EIw''(0,t) - I_{\mathrm{H}}\ddot{\theta} = 0 \tag{2}$$

$$m\left[\ddot{w}(l,t) + l\ddot{\theta}(t)\right] = EIw'''(l,t) \tag{3}$$

$$w(0) = w'(0) = w''(l) = 0 \tag{4}$$

Semigroup Theory

Consider an abstract Cauchy problem,

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0 \tag{5}$$

$$z(0) = z_0 \in D(A) \tag{6}$$

where A is a closed operator with $D(A)$ dense in Z. The solution of (5)-(6) is,

$$z(t) = T(t)z_0 + \int_0^t T(t-s)u(s)ds \tag{7}$$

Definition

Definition 4.1 Let Z be a Hilbert space. A C_0 semigroup of operators is a family of bounded operators $\{T(t), t \geq 0\}$ on Z that satisfies

1. $T(t+s) = T(t)T(s)$
2. $T(0) = I$
3. $\|T(t)z_0 - z_0\| \to 0$ as $t \to 0^+ \quad \forall z_0 \in Z$

Theorem 4.2 A C_0 semigroup $T(t)$ on Z has the following properties:

1. If $w_0 = \inf\left(\frac{1}{t}\log\|T(t)\|\right)$, then $w_0 = \lim_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right)$
2. $\forall w > w_0$ there exists a constant $M > 1, w > 0$ such that $\|T(t)\| \leq Me^{wt} \quad \forall t \geq 0$

Infinitesimal generator & Resolvent Operator

Definition 4.3 The *infinitesimal generator* A of a C_0-semigroup on a Hilbert space Z is defined by

$$Az = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)z$$

$D(A)$ is the set of elements in Z for which the limit exists.

Theorem 4.4 Let $T(t)$ be a C_0 semigroup with infinitesimal generator A and growth bound w_0. If $\mathrm{Re}(\lambda) > \omega > \omega_0$ then $\lambda \in \rho(A)$, and for all $z \in Z$

$$R(\lambda, A)z = (\lambda I - A)^{-1} z = \int_0^\infty e^{-\lambda t} T(t) dt$$

Characterization of infinitesimal generator

Definition 4.5 $T(t)$ is a *contraction semigroup* if $\|T(t)\| < 1 \ , \forall t \geq 0$

Theorem 4.6 Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ are:

$$\mathrm{Re}\,\langle Az, z\rangle \;\leq\; \omega\|z\|^2 \quad \forall z \in D(A) \tag{8}$$

$$\mathrm{Re}\,\langle A^* z, z\rangle \;\leq\; \omega\|z\|^2 \quad \forall z \in D(A^*) \tag{9}$$

Stability

1. $T(t)$ is *asymptotically stable* if

$$\|T(t)z\| \to 0 \quad \text{if} \quad t \to \infty \quad , \quad \forall z \in Z$$

2. $T(t)$ is *exponentially stable* if there exist $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}$$

3. $T(t)$ is *weakly stable* if $\forall x \, \forall y \in Z$

$$\langle T(t)x, y\rangle \to 0 \quad , \quad t \to \infty$$

To prove the asymptotically stability

Theorem 4.7 Let $T(t)$ be a uniformly bounded semigroup on a Banach space X with the infinitesimal generator A and

1. $\sigma(A) \cap i\mathbb{R}$ is countable
2. $\sigma_P(A^*) = \emptyset$

then $T(t)$ is asymptotically stable.

Notation

- $H^m(0,l)$: Sobolev space order m with norm given by

$$\|u\|_{H^m}^2 = \sum_{0\le|\alpha|\le m} \|D^\alpha u\|^2$$

- $H_0^2(0,l)$: $\{u \in H^2(0,l) \mid u(0) = u'(0) = 0\}$ with norm given by

$$\|u\|_{H_0^2}^2 = \|u''\|^2$$

- $C_B^j(0,l)$: $\{u \in C^j(0,l) \mid D^\alpha u \quad \text{is bounded} \quad \}$
- $C^{m,\lambda}(0,l)$: $\{u \in C^m(0,l) \mid |D^\alpha u(x) - D^\alpha u(y)| \le K|x-y|^\lambda\}$

Result : $\|\cdot\|_{H_0^2} \sim \|\cdot\|_{H^2}$

Sobolev Imbedding Theorem

Definition 6.1 Let X and Y be Banach spaces. We say that X is *imbedded* in Y and write $X \to Y$ if

1. X is a subspace of Y, and
2. The identity operator $I : X \to Y$ is continuous. i.e., there exists $M > 0$ such that

$$\|Ix\|_Y \leq M\|x\|_X, \quad \forall x \in X$$

From the Sobolev Imbedding theorem and the Hilbert-Schmidt imbedding theorem, we can list the imbeddings that are used here:

1. $H^4(0,l) \to C_B^3(0,l)$ and $H^2(0,l) \to C_B^1(0,l)$
2. $H^2(0,l) \to C^{0,\lambda}[0,l]$

$$|u(l)| \leq M_1\|u''\| \ \ \forall u \in H_0^2(0,l) \tag{10}$$

3. $I : H^2(0,l) \to L_2(0,l)$ is compact.
   $\Longrightarrow I : H_0^2(0,l) \to L_2(0,l)$ is also compact.

The Closed-Loop System

We apply the control law

$$\tau(t) = -EIw''(0,t) + KI_{\mathrm{H}}\left[\rho\left\langle \dot{w}, x\right\rangle_H + ml\dot{w}(l,t)\right] \tag{11}$$

Substitute (11) in (2), the closed-loop equations are:

$$\ddot{w}(x,t) + \frac{EI}{\rho}w''''(x,t) = -xK\left[\rho\left\langle \dot{w}, x\right\rangle + ml\dot{w}(l,t)\right] \tag{12}$$

$$w(0,t) = w'(0,t) = w''(l,t) = 0 \tag{13}$$

$$m\ddot{w}(x,t) + mlK\left[\rho\left\langle \dot{w}, x\right\rangle + ml\dot{w}(l,t)\right] = EIw'''(l,t) \tag{14}$$

Problem formulation

Let $H = L_2(0,l)$ and consider the Hilbert space $\mathcal{H} = H_0^2(0,l) \oplus L_2(0,l) \oplus \mathbb{C}$ with an inner product

$$\langle u, v \rangle = EI \left\langle u_1'', v_1'' \right\rangle_H + \rho \left\langle u_2, v_2 \right\rangle_H + m \left\langle u_3, v_3 \right\rangle_{\mathbb{C}} \tag{15}$$

we can write (12)-(14) in the form $\dot{z} = \mathcal{A}z$, where

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\frac{EI}{\rho}\frac{\partial^4}{\partial x^4} & -Kx\rho\left\langle \cdot, x \right\rangle & -Kxml \\ \frac{EI}{m}\frac{\partial^3}{\partial x^3}\,|_{x=l} & -Kl\rho\left\langle \cdot, x \right\rangle & -Klml \end{bmatrix} \tag{16}$$

$$D(\mathcal{A}) = \big\{ (z_1, z_2, z_3) \in H^4(0,l) \oplus H_0^2(0,l) \oplus \mathbb{C} \mid$$
$$z_1(0) = z_1'(0) = z_1''(l) = 0, z_2(l) = z_3 \big\}$$

$$z(t) = \big[w(\cdot,t) \;\; \dot{w}(\cdot,t) \;\; \dot{w}(l,t) \big]^T \in \mathcal{H}$$

$\mathcal{A}$ generates a C_0 semigroup

Define the operator $Q : \mathcal{H} \to \mathcal{H}$

$$Qv = \begin{bmatrix} \frac{Kq_2(x)}{EI}\left[\rho\left\langle v_1, x\right\rangle + mlv_1(l)\right] - \frac{\rho}{EI}\int_0^x\int_0^{x_4}\int_{x_3}^l\int_{x_2}^l v_2(x_1)dx_1dx_2dx_3dx_4 + \frac{mq_1(x)}{EI}v_3 \\ v_1(x) \\ v_1(l) \end{bmatrix} \quad (17)$$

where

$$\begin{aligned} q_1(x) &= \frac{x^3}{6} - \frac{lx^2}{2} \\ q_2(x) &= \rho\left(\frac{l^2x^3}{12} - \frac{l^3x^2}{6} - \frac{x^5}{120}\right) + mlq_1(x) \end{aligned}$$

Lemma 8.1

1. Q is the inverse of $\mathcal{A}$
2. $\mathcal{A}^{-1}$ is a bounded operator.

Note : $D(Q) = \mathcal{H} = \mathcal{R}(\mathcal{A})$. $\Longrightarrow \mathcal{A}$ is onto.

Theorem 9.1 (Closed graph Theorem) Let X, Y be Banach spaces. A linear operator $T : X \to Y$ is bounded if and only if T is closed.

Therefore, $\mathcal{A}^{-1}$ is closed. $\Longrightarrow \mathcal{A}$ is closed.

Definition 9.2 The resolvent set of a closed linear operator A is

$$\rho(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \quad \text{is bijective} \quad \}$$

Result : $\Longrightarrow 0 \in \rho(\mathcal{A})$

The Adjoint operator $\mathcal{A}^*$

From the definition of the adjoint operator, we have

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I & 0 \\ \frac{EI}{\rho}\frac{\partial^4}{\partial x^4} & -Kx\rho\langle\cdot, x\rangle & -Kxml \\ -\frac{EI}{m}\frac{\partial^3}{\partial x^3}\,|_x = l & -K\rho l\langle\cdot, x\rangle & -Klml \end{bmatrix} \qquad (18)$$

$$D(\mathcal{A}^*) = \{(v_1, v_2, v_3) \in H^4(0,l) \oplus H_0^2(0,l) \oplus \mathbb{C} \mid v_2(0) = v_2'(0) = v_1''(l) = 0, v_3 = v_2(l)\}$$

Theorem 9.3 $\mathcal{A}$ generates a contraction semigroup.

proof. From the calculation,

$$\mathrm{Re}\,\langle \mathcal{A}u, u\rangle_{\mathcal{H}} = -K\left|\rho\,\langle u_2, x\rangle + mlu_3\right|^2 \le 0 \tag{19}$$

$$\mathrm{Re}\,\langle \mathcal{A}^{*}u, u\rangle_{\mathcal{H}} = -K\left|\rho\,\langle u_2, x\rangle + mlu_3\right|^2 \le 0 \tag{20}$$

The equations (8)-(9) are satisfied with $\omega = 0$ □

Stability Analysis

- The spectrum of the infinitesimal generator
- Eigenvalue analysis
- Closed-loop stability

The spectrum of the infinitesimal generator

To prove that the spectrum set consists of only the eigenvalues

Theorem 10.1 Let A be a closed linear operator with $0 \in \rho(A)$ and A^{-1} compact. The spectrum of A consists of only isolated eigenvalues with finite multiplicity.

■

Lemma 10.2 $\mathcal{A}^{-1}$ is compact.

Proof. $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ can be written in the following form,

$$\mathcal{A}^{-1} = \begin{bmatrix} T_1 & T_2 & T_3 \\ I & 0 & 0 \\ T_4 & 0 & 0 \end{bmatrix}$$

We will prove the compactness property of each T_i as follows:

1. Consider $T_1 : H_0^2(0,l) \to H_0^2(0,l)$ defined by

$$T_1 v = \frac{K}{EI} q_2(x)(\rho \langle v, x \rangle + mlv(l))$$

Let S_N be a bounded set of $v \in H_0^2(0,l)$ with $\|v\|_{H_0^2} \le N$. Then,

$$\begin{aligned}
\|T_1 v\|_{H_0^2} &= \frac{K}{EI}\|q_2(x)\|_{H_0^2}|\rho \langle v, x \rangle + mlv(l)| \\
&\le \frac{K}{EI}\|q_2(x)\|_{H_0^2}(\rho|\langle v, x \rangle| + ml|v(l)|) \\
&\le \frac{K}{EI} ml^2 \left\{ \frac{\rho l}{\sqrt{2}} \|v\|_{L_2} + mlM_1\|v\|_{H_0^2} \right\} \qquad (21) \\
&\le \frac{Kml^2}{EI} \left\{ \frac{\rho l}{\sqrt{2}} N' + mlM_1 N \right\} \qquad (22) \\
&\le M_2
\end{aligned}$$

(using (10), the Cauchy-schwartz ineq., and $\| \cdot \|_{H^2} \sim \| \cdot \|_{H_0^2}$)

This shows that the image of T_1 is uniformly bounded.

Since $q_2(x)$ is continuous , i.e.,

$$\forall x_0 \in (0,l), \forall \epsilon_1 > 0, \quad \exists \delta_1 > 0 \ \text{ s.t.}$$

$$\|q_2(x) - q_2(x_0)\| < \epsilon_1, \ \text{ whenever } \ |x - x_0| < \delta_1$$

$$\begin{aligned} \|T_1 v(x) - T_1 v(x_0)\| &= \frac{K}{EI} |\rho \langle v, x \rangle + mlv(l)| \|q_2(x) - q_2(x_0)\| \\ &\leq \frac{K}{EI} \left\{ \frac{\rho l}{\sqrt{2}} N' + mlM_1 N \right\} \|q_2(x) - q_2(x_0)\| \end{aligned}$$

Let $\epsilon = EI\epsilon_1 / K(\frac{\rho l}{\sqrt{2}} N' + mlM_1 N)$, so

$$\|T_1 v(x) - T_1 v(x_0)\| < \epsilon \ \text{ whenever } \ |x - x_0| < \delta_1$$

Note: δ_1 does not depend on the choice of $v \in S_N \Longrightarrow$ the image of T_1 is equicontinuous.

Theorem 10.3 (Arzela's theorem) Let Ω be a bounded domain in $\mathbb{R}$. A subset K of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ provided that

1. K is uniformly bounded. i.e., there exists a constant M such that

$$\forall \phi \in K, x \in \Omega, \quad |\phi(x)| \le M$$

2. K is equicontinuous. i.e., $\forall \epsilon > 0, \exists \delta > 0$ such that if $\phi \in K, x, y \in \Omega$, and $|x - y| < \delta$ then $|\phi(x) - \phi(y)| < \epsilon$.

The image of T_1 is a precompact set $\Longrightarrow T_1$ is compact.

2. Consider $T_2 : L_2(0,l) \to H_0^2(0,l)$ defined by

$$T_2 v = -\frac{\rho}{EI}\int_0^x \int_0^{x_4} \int_{x_3}^l \int_{x_2}^l v(x_1)dx_1dx_2dx_3dx_4$$

Let $f \in L_2(0,l)$ and let χ_S be the characteristic function of a set S. Then

$$\chi_{(0,x)} \in L_2[0,l] \times L_2[0,l]$$

because $\int_{[0,l]} \int_{[0,l]} \chi_{(0,x)} dxdy = xl < \infty$,$0 \leq x \leq l$.

Thus the operator A defined by

$$Af = \int_0^x f(\tau)d\tau = \int_0^l \chi_{(0,x)} f(\tau)d\tau$$

is a compact operator from $L_2(0,l) \to L_2(0,l) \Longrightarrow T_2$ is compact.

3. Consider $T_3 : \mathbb{C} \to H_0^2(0,l)$ defined by

$$T_3 v = \frac{m}{EI} q_1(x) v$$

As in the case of T_1, we can see that T_3 is compact.

4. The imbedding mapping from $H_0^2(0,l) \to L_2(0,l)$ is compact. This follows from the Hilbert-Schmidt imbedding theorem

5. $T_5 : H_0^2(0,l) \to \mathbb{C}$, $T_5 v = v(l)$

From (10), T_5 is a bounded linear functional. Its image has a finite dimensional range. $\Longrightarrow T_5$ is compact.

From 1-5, we can conclude that $\mathcal{A}^{-1}$ is compact. □

Now we have,

- $0 \in \rho(\mathcal{A})$
- $\mathcal{A}^{-1}$ is compact.
- From theorem 10.1, the spectrum of $\mathcal{A}$ consists of only isolated eigenvalues with finite multiplicity.

The eigenvalues

Let λ and $\phi(x) = \left[\phi_1(x) \;\; \phi_2(x) \;\; \phi_3\right]^T$ be an eigenvalue and the corresponding eigenvector of $\mathcal{A}$.

$$\mathcal{A}\phi(x) = \lambda\phi(x) \tag{23}$$

The eigenvalues are the solutions of,

$$\frac{\rho K(\mathrm{sh}\cdot\mathrm{c} - \mathrm{ch}\cdot\mathrm{s}) - 2Kml\beta\cdot\mathrm{sh}\cdot\mathrm{s}}{\beta^2(\lambda + \frac{\rho K l^3}{3} + Kml^2)} + \beta\left\{1 + \mathrm{ch}\cdot\mathrm{c} + \frac{m\beta}{\rho}(\mathrm{sh}\cdot\mathrm{c} - \mathrm{ch}\cdot\mathrm{s})\right\} = 0 \tag{24}$$

where

$$\mathrm{s} \equiv \sin(\beta l) \qquad \mathrm{c} \equiv \cos(\beta l) \qquad \mathrm{sh} \equiv \sinh(\beta l) \qquad \mathrm{ch} \equiv \cosh(\beta l)$$

$$\lambda = -i\beta^2\sqrt{\frac{EI}{\rho}}$$

Next, we will show that all eigenvalues lie in the open LHP.

Eigenvalue Analysis

Lemma 10.4 Consider the following equations,

$$h_1(\beta) = \sinh(\beta l) + \sin(\beta l) = 0 \tag{25}$$

$$h_2(\beta) = \cosh(\beta l) + \cos(\beta l) + k\beta(\sinh(\beta l) - \sin(\beta l)) = 0 \tag{26}$$

where $k > 0$ is a constant. If $\beta = a + ib$ is a solution of either (25) or (26) then $|a| = |b|$. Moreover, (25) and (26) have distinct solutions.

If the solution β satisfies $|a| = |b|$, equation (25)-(26) can be rewritten as

$$h_1(\beta) = 0 \Longleftrightarrow h_{1a}(a) = \cos(al)\sinh(al) + \sin(al)\cosh(al) = 0 \tag{27}$$

$$h_2(\beta) = 0 \Longleftrightarrow h_{2a}(a) = \cos(al)\cosh(al) + ka(\cos(al)\sinh(al) - \sin(al)\cosh(al)) = 0 \tag{28}$$

Let a_0 be a solution of (27), then

$$\sin(a_0 l)\cosh(a_0 l) = -\cos(a_0 l)\sinh(a_0 l)$$

Substitute in (28) we get

$$h_{2a}(a_0) = \cos(a_0 l)[\cosh(a_0 l) + 2ka_0 \sinh(a_0 l)]$$

Since $\cos(a_0 l) \neq 0$ and

$$a_0 > 0 \Rightarrow \sinh(a_0 l) > 0 \;\Rightarrow\; a_0 \sinh(a_0 l) > 0$$
$$a_0 < 0 \Rightarrow \sinh(a_0 l) < 0 \;\Rightarrow\; a_0 \sinh(a_0 l) > 0$$

therefore,

$$\cosh(a_0 l) + 2ka_0 \sinh(a_0 l) > 0 \quad \forall a_0 \in \mathbb{R}$$

$\Longrightarrow$ If a_0 is a solution of $h_1(a) = 0$, then $h_2(a_0) \neq 0$, i.e., they have no common solutions. $\square$

Lemma 10.5 Let λ and $\phi(x) = \left[\phi_1(x) \;\; \lambda\phi_1(x) \;\; \lambda^2\phi_1(l)\right]^T$ be an eigenvalue and the corresponding eigenvector of $\mathcal{A}$ respectively. Then,

$$\rho\left\langle \phi_1, x\right\rangle + ml\phi_1(l) \neq 0$$

Proof. Assume $F(\phi_1) \equiv \rho\left\langle \phi_1, x\right\rangle + ml\phi_1(l) = 0$. From $\mathcal{A}\phi(x) = \lambda\phi(x)$, we can find $\phi_1(x)$

$$\phi_1(x) = c_1(\cosh(\beta x) - \cos(\beta x)) + c_3(\sinh(\beta x) - \sin(\beta x))$$

where c_1, c_3 satisfy

$$c_1(\mathrm{ch} + \mathrm{c}) + c_3(\mathrm{sh} + \mathrm{s}) = 0 \tag{29}$$

$$c_1\left\{(\mathrm{sh} - \mathrm{s}) + \frac{m\beta}{\rho}(\mathrm{ch} - \mathrm{c})\right\} + c_3\left\{(\mathrm{ch} + \mathrm{c}) + \frac{m\beta}{\rho}(\mathrm{sh} - \mathrm{s})\right\} = 0 \tag{30}$$

$$c_1\left\{\rho l\beta(\mathrm{sh} - \mathrm{s}) - \rho(\mathrm{ch} + \mathrm{c}) + 2\rho + ml\beta^2(\mathrm{ch} - \mathrm{c})\right\} + c_3\left\{\rho l\beta(\mathrm{ch} + \mathrm{c}) - \rho(\mathrm{sh} + \mathrm{s}) + ml\beta^2(\mathrm{sh} - \mathrm{s})\right\} = 0 \tag{31}$$

or,

$$\begin{bmatrix} (\mathrm{ch}+\mathrm{c}) & (\mathrm{sh}+\mathrm{s}) \\ (\mathrm{sh}-\mathrm{s})+\frac{m\beta}{\rho}(\mathrm{ch}-\mathrm{c}) & (\mathrm{ch}+\mathrm{c})+\frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s}) \\ \rho l\beta(\mathrm{sh}-\mathrm{s})-\rho(\mathrm{ch}+\mathrm{c})+2\rho+ml\beta^2(\mathrm{ch}-\mathrm{c}) & \rho l\beta(\mathrm{ch}+\mathrm{c})-\rho(\mathrm{sh}+\mathrm{s})+ml\beta^2(\mathrm{sh}-\mathrm{s}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

using row operation,

$$\begin{bmatrix} (\mathrm{ch}+\mathrm{c}) & (\mathrm{sh}+\mathrm{s}) \\ (\mathrm{sh}-\mathrm{s})+\frac{m\beta}{\rho}(\mathrm{ch}-\mathrm{c}) & (\mathrm{ch}+\mathrm{c})+\frac{m\beta}{\rho}(\mathrm{sh}-\mathrm{s}) \\ 2\rho & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

➜ $c_1 = 0$

➜ From lemma 10.4 $\Longrightarrow c_3 = 0$

➜ $\phi_1(x) = 0 \Longrightarrow \phi(x)$ is not the eigenvector of $\mathcal{A}$,

which is a contradiction □

From the eigenvalue problem,

$$\phi_1''''(x) + \frac{\rho\lambda^2}{EI}\phi_1(x) = -\frac{\rho K}{EI}\lambda\left[\rho\left\langle \phi_1, x\right\rangle + ml\phi_1(l)\right]\cdot x \tag{32}$$

$$\phi_1(0) = \phi_1'(0) = \phi_1''(l) = 0 \tag{33}$$

$$\phi_1'''(l) = \frac{Kml}{EI}\lambda\left[\rho\left\langle \phi_1, x\right\rangle + ml\phi_1(l)\right] + \frac{m}{EI}\lambda^2\phi_1(l) \tag{34}$$

Take the inner product with ϕ_1 on both sides in (32)

$$\left\langle \phi_1'''', \phi_1\right\rangle + \frac{\rho\lambda^2}{EI}\left\langle \phi_1, \phi_1\right\rangle + \frac{\rho K\lambda}{EI}\left(\rho\left\langle \phi_1, x\right\rangle + ml\phi_1(l)\right)\left\langle x, \phi_1\right\rangle = 0 \tag{35}$$

since

$$\left\langle \phi_1'''', \phi_1\right\rangle = \lambda\frac{\rho Kml}{EI}\left\langle \phi_1, x\right\rangle\overline{\phi_1(l)} + \lambda\frac{Km^2l^2}{EI}|\phi_1(l)|^2 + \lambda^2\frac{m}{EI}|\phi_1(l)|^2 + \|\phi''\|^2 \tag{36}$$

substitute in (35), we get

$$\lambda^2\left\{m|\phi_1(l)|^2+\rho\|\phi_1\|^2\right\}+\lambda K\left|\rho\left\langle\phi_1,x\right\rangle+ml\phi_1(l)\right|^2+EI\|\phi''\|^2=0 \tag{37}$$

Let $\lambda = a + ib$, (37) can be split into two equations.

$$(a^2-b^2)(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+a\cdot K\left|\rho\left\langle\phi_1,x\right\rangle+ml\phi_1(l)\right|^2+EI\|\phi''\|^2=0 \tag{38}$$

$$2ab(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+b\cdot K\left|\rho\left\langle\phi_1,x\right\rangle+ml\phi_1(l)\right|^2=0 \tag{39}$$

If $b = 0$, from (38)

$$a^2(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)+a\cdot K\left|\rho\left\langle\phi_1,x\right\rangle+ml\phi_1(l)\right|^2+EI\|\phi''\|^2=0$$

From lemma 10.5, the coefficients of the polynamial in the variable a are all positive. Thus $a < 0$.

If $b \neq 0$ from (39)

$$a=-\frac{K\left|\rho\left\langle\phi_1,x\right\rangle+ml\phi_1(l)\right|^2}{2(m|\phi_1(l)|^2+\rho\|\phi_1\|^2)}<0$$

Thus $\mathrm{Re}(\lambda) < 0$.

Closed-Loop Stability

- ✔ $\sigma(\mathcal{A}) = \sigma_P(\mathcal{A})$
- ✔ The real part of all eigenvalues are negative.
- ✔ $\sigma(\mathcal{A}) \cup i\mathbb{R} \Longrightarrow$ is countable.
- ✔ $\sigma_P(\mathcal{A}^*) = \sigma_r(\mathcal{A}) = \emptyset$
- ✔ A contraction semigroup is uniformly bounded.
- ✔ From theorem 4.7, the semigroup is asymptotically stable.

Conclusions

- Feedback control signal through motor acceleration.
- The Proposed control law is the sum of the tip deflection and its linear functional.
- The infinitesimal generator of the closed-loop system generates a contractions semigroup.
- The spectrum consists of only the eigenvalues.
- All eigenvalues have negative real parts.
- The closed-loop system is asymptotically stable.