Exploring Granger Causality for Time series via Wald Test on Estimated Models with Guaranteed Stability

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Nov 3, 2016

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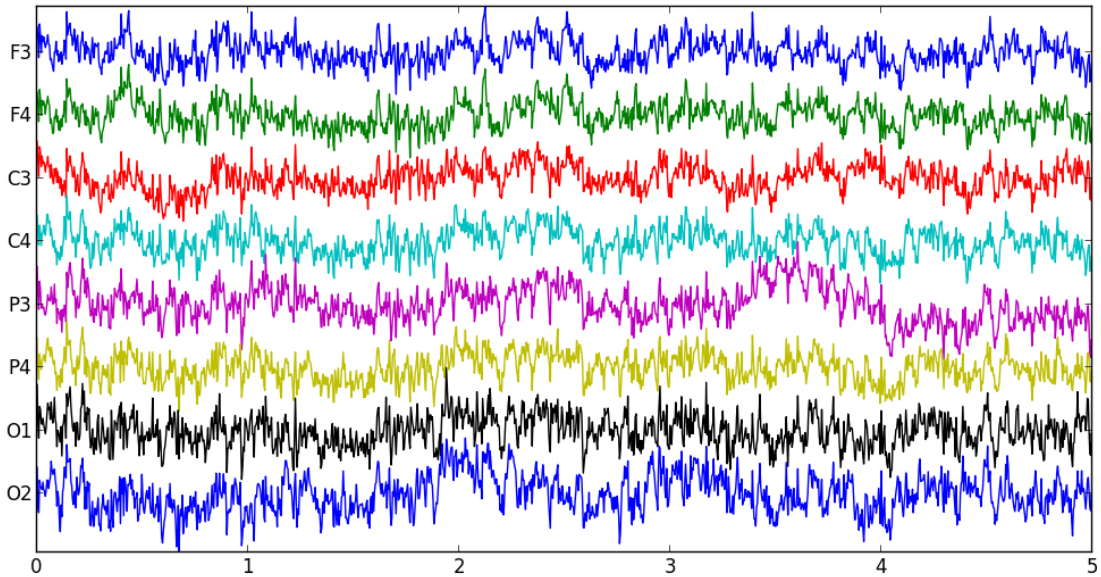
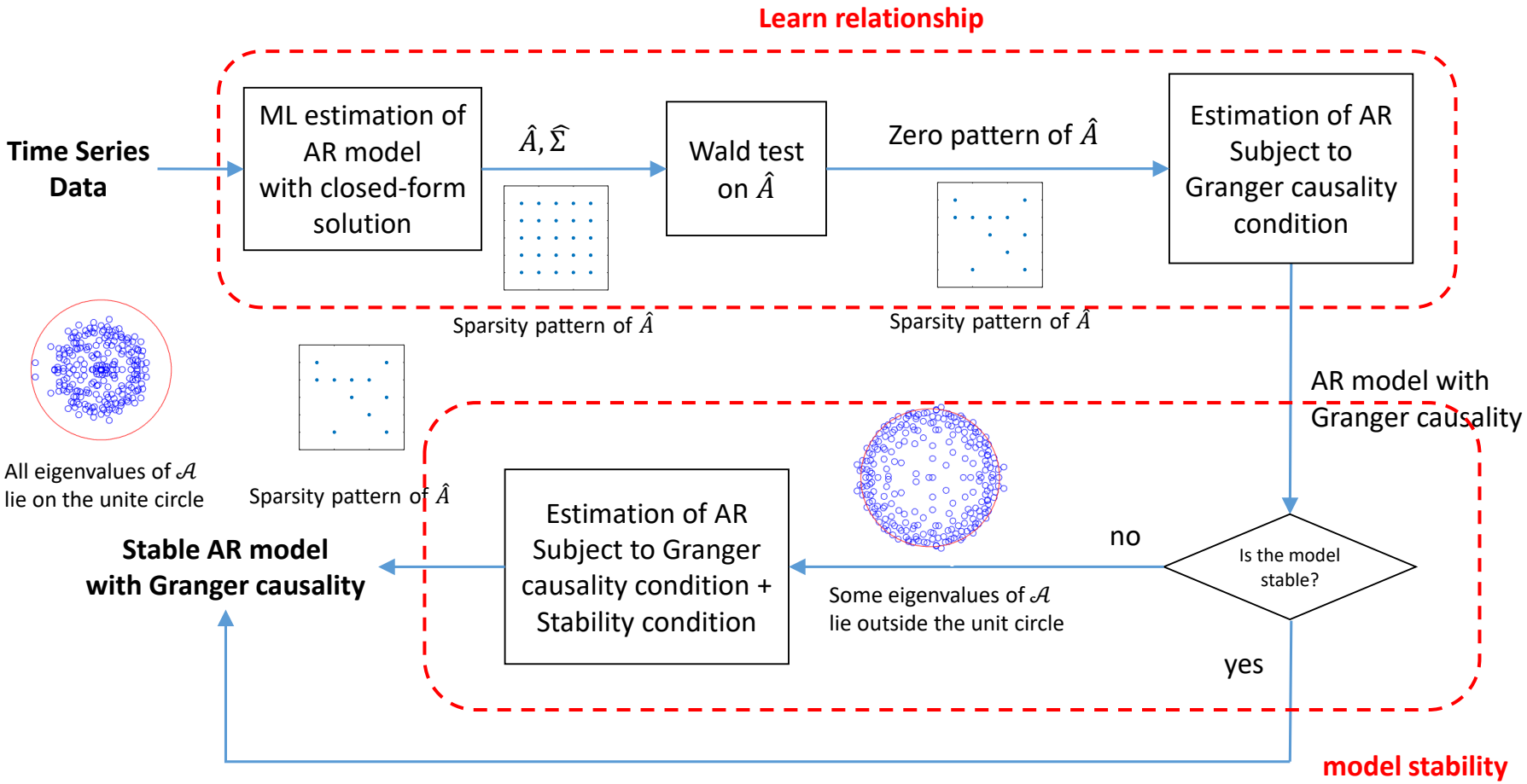
Introduction and Objective



By knowing the relationship of the parameters in time series data, we can explain the dynamic of the time series data.

- explain relationship between the time series data by using the Granger causality concept and autoregressive model.
- estimate stable model with Granger causality constraints.

Flow Chart of the Paper



Auto-Regressive (AR) Model

Time series data can be represented by an AR model,

$$y(t) = c + A_1 y(t-1) + A_2 y(t-2) + ... + A_p y(t-p) + v(t) \qquad (1)$$

- $y(t) = (y_1(t), y_2(t), ..., y_n(t)) \in \mathbb{R}^n$
- $A_1, A_2, ..., A_p \in \mathbb{R}^{n \times n}$ are AR coefficients (p is lag order of the model)
- $c \in \mathbb{R}^n$ is a constant vector
- $v(t)$ is a Gaussian noise process with variance Σ
- the observations $y(1), y(2), ..., y(N)$ are available.
- $y(1), y(2), ..., y(p)$ are deterministic values and given.

Granger causality

"Granger causality" is a term for a specific notion of causality in time-series analysis. The idea of Granger causality is a simple one:

$$X \xrightarrow{G-causes} Y$$

A variable X "Granger-causes" Y if Y can be better predicted using the histories of both X and Y than it can using the history of only Y.

Granger Causality

- Apply the concept of Granger causality to AR model in equation (1)

the causality of the model can be written in linear equation form that is if y_j "not Granger-cause" to y_i then

$$(A_k)_{ij} = 0, \qquad k = 1, 2, .., p$$

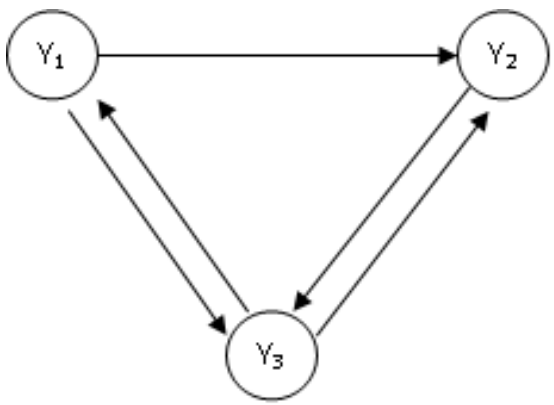
where $(A_k)_{ij}$ denotes the (i, j) entry of A_k.

- Granger Causality structure can be read from the zero pattern of estimated AR coefficient matrix.

example

Consider AR(4) (AR model when p=4) and $y(t) \in \mathbb{R}^3$,
if y_2 "not Granger-cause" to y_1 then the model could be

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = c + \begin{bmatrix} X & 0 & X \\ X & X & X \\ X & X & X \end{bmatrix} \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ y_3(t-1) \end{bmatrix} + \begin{bmatrix} X & 0 & X \\ X & X & X \\ X & X & X \end{bmatrix} \begin{bmatrix} y_1(t-2) \\ y_2(t-2) \\ y_3(t-2) \end{bmatrix}$$
$$+ \begin{bmatrix} X & 0 & X \\ X & X & X \\ X & X & X \end{bmatrix} \begin{bmatrix} y_1(t-3) \\ y_2(t-3) \\ y_3(t-3) \end{bmatrix} + \begin{bmatrix} X & 0 & X \\ X & X & X \\ X & X & X \end{bmatrix} \begin{bmatrix} y_1(t-4) \\ y_2(t-4) \\ y_3(t-4) \end{bmatrix} + v(t)$$



Maximum likelihood estimation

To estimate A and Σ by using maximum likelihood estimation, we solving the problem

$$\underset{A,\Sigma}{\text{maximize}} \quad \frac{N-p}{2}\log\det\Sigma^{-1} - \frac{1}{2}\|L(Y-AH)\|_F^2 \tag{2}$$

when $L^TL = \Sigma^{-1}$ and the problem is equivalent to the least-squares problem

$$\underset{A}{\text{minimize}} \quad \|Y-AH\|_F^2 \tag{3}$$

where

$$Y = \begin{bmatrix} y(p+1) & y(p+2) & \cdots & y(N) \end{bmatrix}_{n\times(N-p)},$$

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y(p) & y(p+1) & \cdots & y(N-1) \\ y(p-1) & y(p) & \cdots & y(N-2) \\ \vdots & \vdots & & \vdots \\ y(1) & y(2) & \cdots & y(N-p) \end{bmatrix}_{(np+1)\times(N-p)}$$

Statistical test for Granger causality Analysis

The null hypothesis for Granger causality condition will be

$$H_0 : (A_k)_{ij} = 0 \quad \text{for} \quad k = 1, 2, \ldots, p$$

and the Wald test is based on the following test statistic:

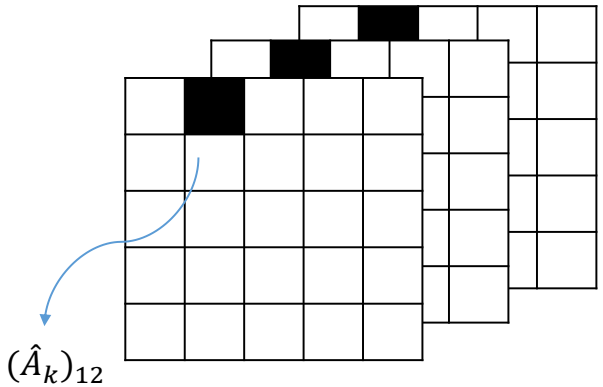
$$W_{ij} = \hat{B}_{ij}^T \left[\widehat{\mathbf{Avar}}(\hat{\theta})_{ij} \right]^{-1} \hat{B}_{ij}$$

where $\hat{B}_{ij} = \left((\hat{A}_1)_{ij}, (\hat{A}_2)_{ij}, \ldots, (\hat{A}_p)_{ij} \right)$, $\hat{\theta}$ is the vectorization of $\hat{A}$, and $\widehat{\mathbf{Avar}}(\hat{\theta})_{ij}$ is the main diagonal block of a consistent estimate of the asymptotic covariance matrix of $\hat{\theta}$.

Statistical test for Granger causality Analysis

IDEA : if H_0 is true, $(\hat{A}_k)_{ij}$ should equal to 0
In Wald test, H_0 is reject if W > C

where C = $F^{-1}(1-\alpha)$ is critical value
and α = **Prob**$(W > C)$ is the significance level.



If $W_{ij} > C$ ⟶ Non Zero

C = 9.4877 when $\alpha = 0.05$ and $p = 3$

W

X	3.2486	35.0241	0.1986	11.5292
0.1279	X	153.831	35.5729	3.2287
13.4569	0.608	X	15.0093	2.1324
12.3471	21.6202	30.7289	X	2.2625
43.4160	2.3035	2.2536	1.5259	X

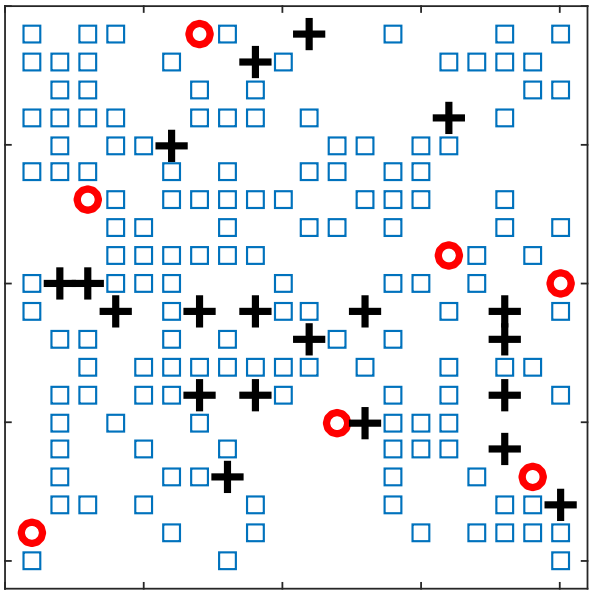
Zeros Structure

X	0	X	0	X
0	X	X	X	0
X	0	X	X	0
X	X	X	X	0
X	0	0	0	X

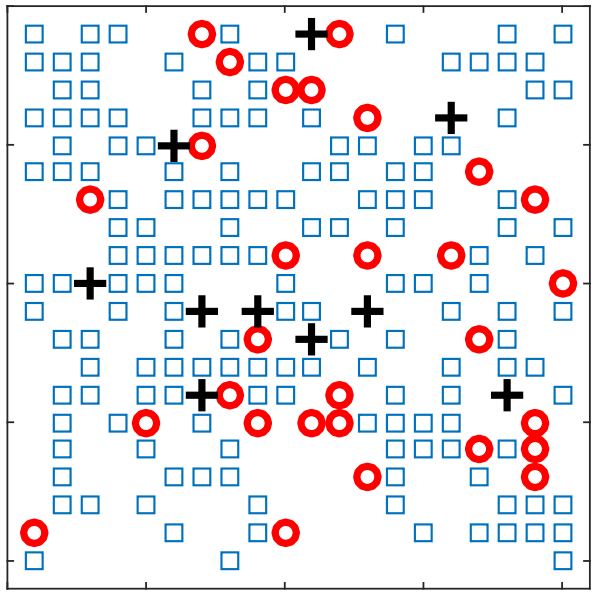
Under the null hypothesis that $(A_k)_{ij} = 0$, the Wald statistic W converges in distribution to Chi-square distribution with p degrees of freedom.

Wald test Results

By generating $y(t) = AH(t) + v(t)$ and the model parameter A_k are generated by choosing some element to be equal to zero.



(a) $\alpha = 0.01$



(b) $\alpha = 0.1$

□ is intersect between non-zeros components
◯ is true component is zero but the estimated is non-zero
$+$ is true component is non-zero but the estimated is zero
and blank is intersect between zero components

AR estimation with Granger causality constraints

After knowing the Granger causality pattern, we solve this optimization problem to estimate A with Granger causality condition.

$$\begin{aligned} \underset{A}{\text{minimize}} \quad & \|Y - AH\|_F^2 \\ \text{subject to} \quad & (A_k)_{ij} = 0 \end{aligned}$$

- The estimated model parameter is not guarantee to be stable
- we need a stability condition.

Stability Condition

we can write the AR model in discrete-time linear system

$$\begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-p+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_2 & \dots & A_{p-1} & A_p \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} y(t-1) \\ y(t-2) \\ \vdots \\ y(t-p+1) \\ y(t-p) \end{bmatrix}$$

The system is stable if and only if $\max_i |\lambda_i(\mathcal{A})| < 1$. The characteristic polynomial have A_k as a coefficient so the condition will be nonlinear in A

Previous Works of Stability condition

Researcher	**Method**	**Difference in paper**
J. Mari et al., 2000 S. L. Lacy et al., 2003	Apply Lyapunov theory to vector ARMA	Too complicate
V. Cerone et al., 2010	Jury's test to guarantee BIBO stability on SISO LTI system	MIMO linear system
L. Buesing et al., 2012	Apply Lyapunov theory on LDS model to estimate A that comply with $A^T A \prec I$	structure of A
K. Turksoy et al., 2013	Use Gershgorin circle theory on ARMAX	similar

Sufficient Condition for stability

Spectral radius and Induced norm

From spectral radius $\rho(\mathcal{A}) = \max_i |\lambda_i(\mathcal{A})|$ then the system is stable if $\rho(\mathcal{A}) < 1$ and by the inequality

$$\rho(\mathcal{A}) \leq \|\mathcal{A}\|$$

if assume that $\|\mathcal{A}\| < 1$ it will affect $\rho(\mathcal{A}) < 1$ when $\|\mathcal{A}\|$ is a induced norm

We choose the infinity-norm of $\mathcal{A}$ to be the sufficient condition for stability.

$$\|\mathcal{A}\|_\infty \leq 1$$

Due to structure of $\mathcal{A}$

- $\|\mathcal{A}\|_1 \leq 1$ and $\|\mathcal{A}\|_2 \leq 1$ lead to meaningless which is $A_1, \ldots, A_{p-1}$ are equal to zero.
- $\|\mathcal{A}\|_F \leq 1$ is impossible

We can estimate a stable model parameter with Granger causality condition by solving this problem.

$$\underset{A}{\text{minimize}} \quad \|Y - AH\|_F^2$$

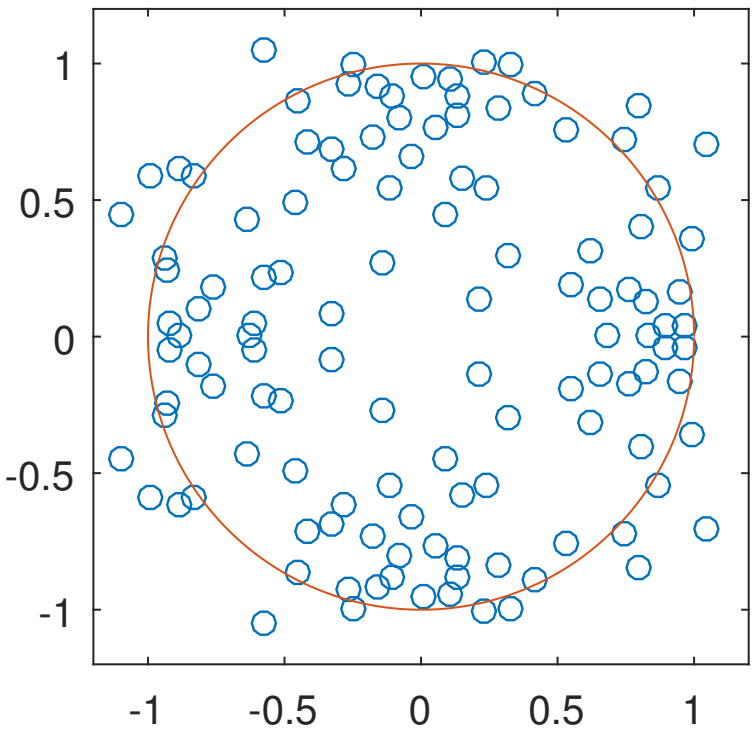
$$\text{subject to} \quad \mathcal{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{p-1} & A_p \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix},$$

$$\|\mathcal{A}\|_\infty \leq 1,$$

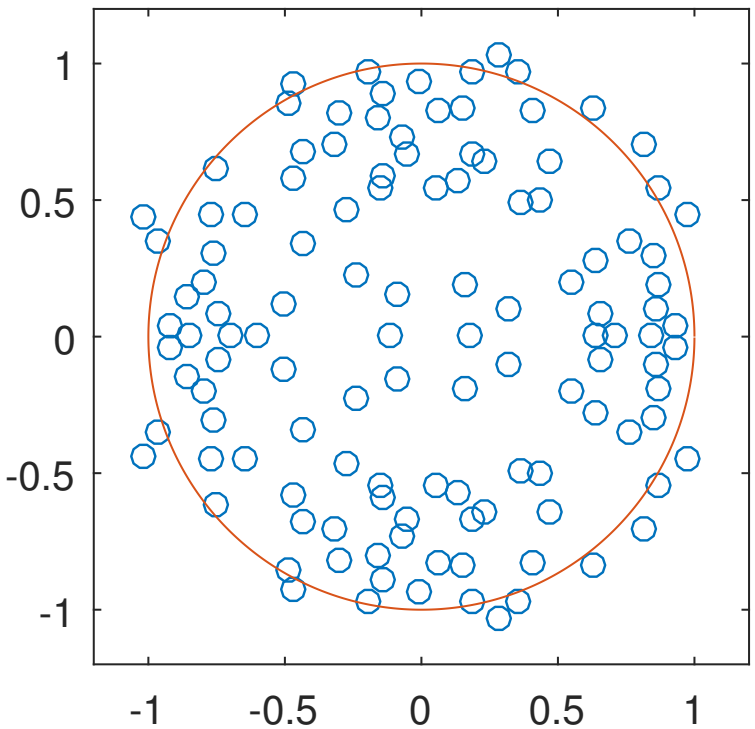
$$(A_k)_{ij} = 0, \qquad (i,j) \subset \{1,\dots,n\}x\{1,\dots,n\}$$

The problem is convex in quadratic form and can be solved by many solver.

Results



(c) no constraint



(d) with Granger causality constraints

Figure 1: Positions of the eigenvalues of $\mathcal{A}$ on complex plane

Results

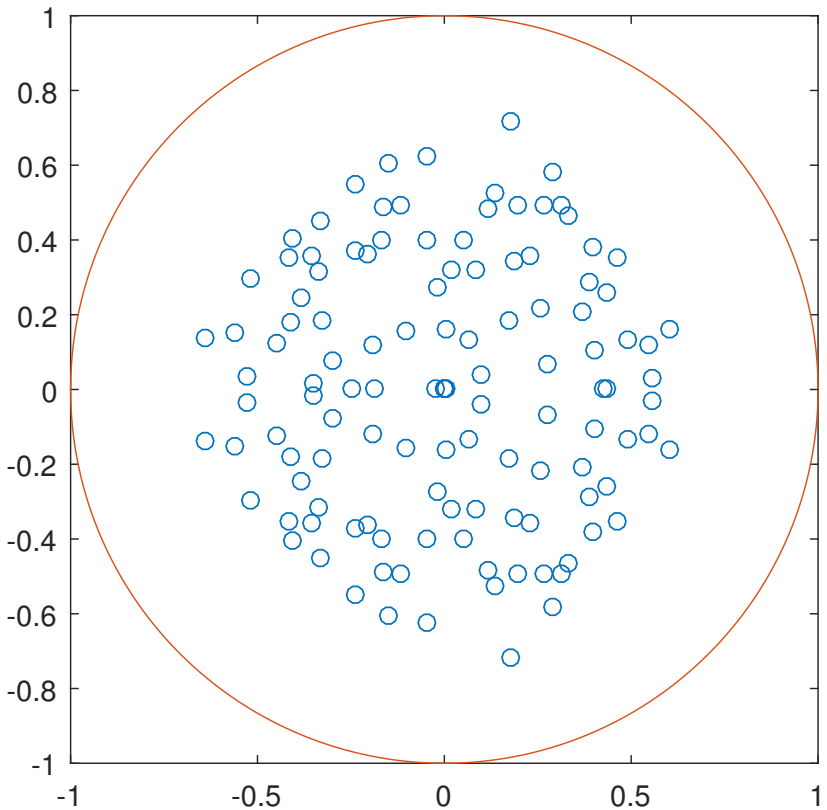
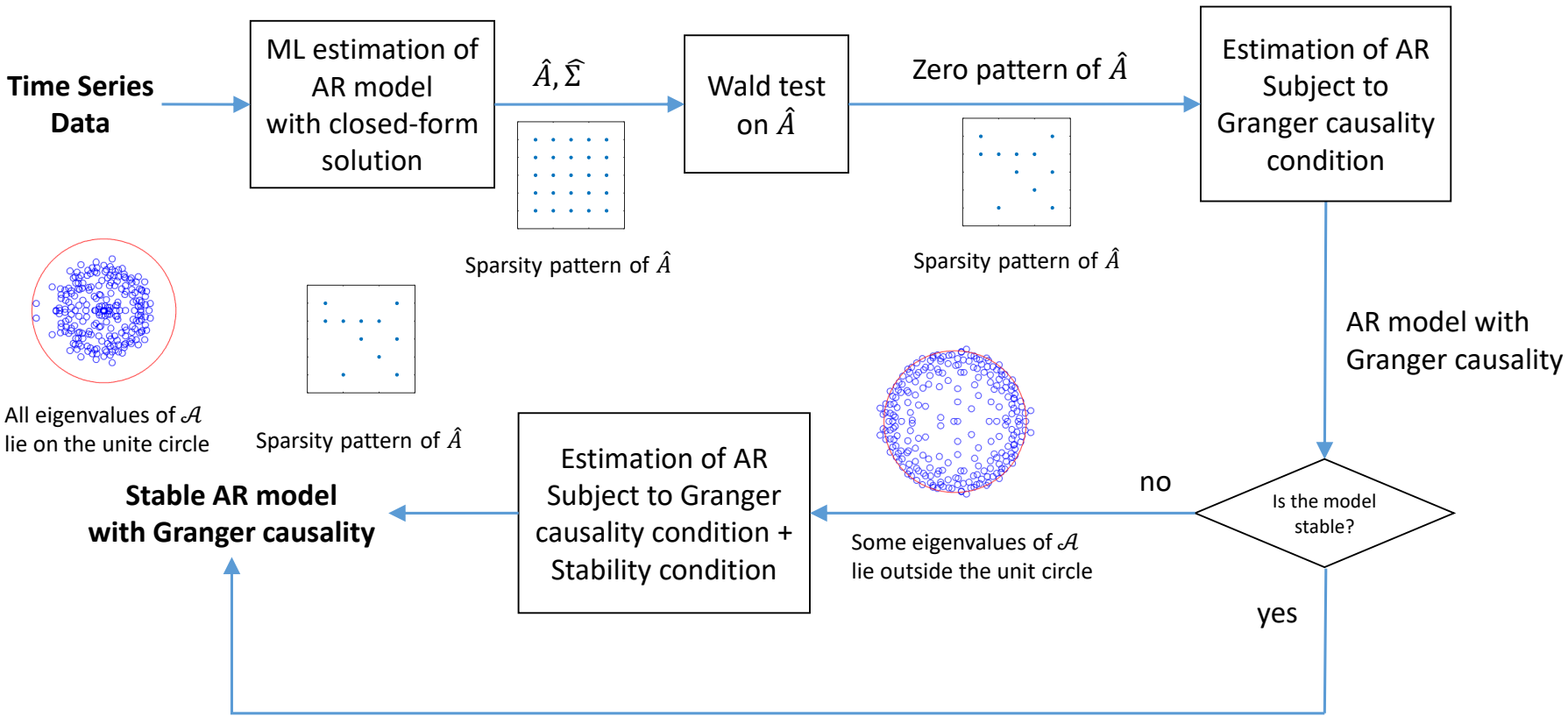
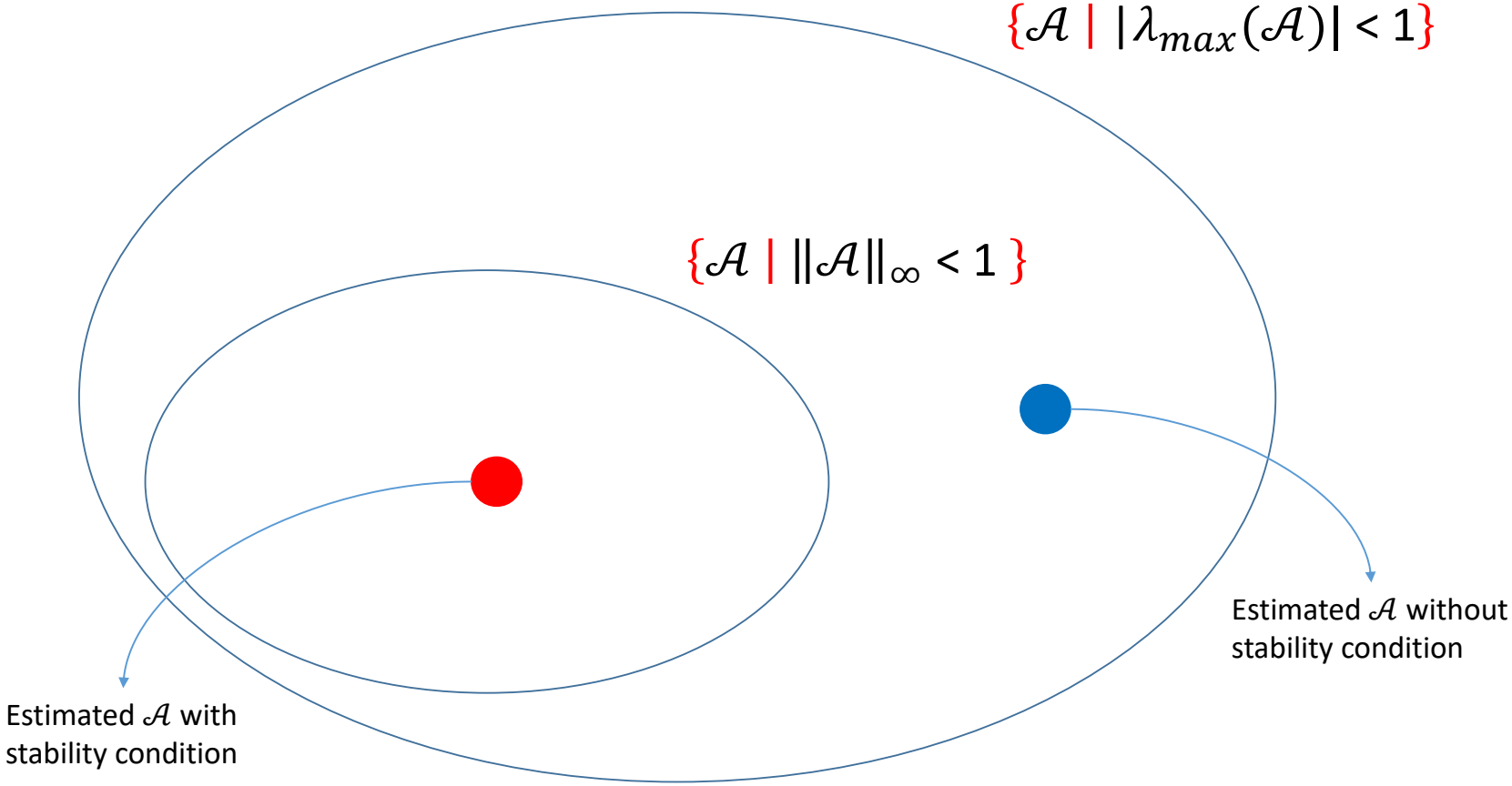


Figure 2: Positions of the eigenvalues of $\mathcal{A}$ on complex plane with Granger causality and stability constraints

Conclusion


Time Series Data
ML estimation of AR model with closed-form solution
$\hat{A}, \hat{\Sigma}$
Wald test on $\hat{A}$
Zero pattern of $\hat{A}$
Estimation of AR Subject to Granger causality condition
Sparsity pattern of $\hat{A}$
Sparsity pattern of $\hat{A}$
AR model with Granger causality
All eigenvalues of $\mathcal{A}$ lie on the unite circle
Sparsity pattern of $\hat{A}$
Is the model stable?
no
Some eigenvalues of $\mathcal{A}$ lie outside the unit circle
Estimation of AR Subject to Granger causality condition + Stability condition
Stable AR model with Granger causality
yes

Why we have to check the stability?



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Q&A

THANK YOU